University of California

Santa Barbara

On Minimal Models and Canonical Models of Elliptic Fourfolds with Section

A dissertation submitted in partial satisfaction

of the requirements for the degree

Doctor of Philosophy

in

Mathematics

by

David Wen

Committee in charge:

Professor David R. Morrison, Chair

Professor Mihai Putinar

Professor Ken Goodearl

June 2018

The Dissertation of David Wen is approved.

Professor Mihai Putinar

Professor Ken Goodearl

Professor David R. Morrison, Chair

June 2018

On Minimal Models and Canonical Models of Elliptic Fourfolds with Section

by

David Wen

To those who believed in me

and to those who didn't

for motivating me to prove them wrong.

Acknowledgements

I would like to thank my advisor, Professor David R. Morrison, for his advice, guidance and encouragement throughout my graduate studies and research. Additionally, I would like to thank the UCSB math department and my colleagues in the department for their support, advice and conversations.

Curriculum Vitae

David Wen

dwen@math.ucsb.edu

Department of Mathematics
South Hall, Room 6607
University of California
Santa Barbara, CA 93106-3080
Citizenship: U.S.A.

Education

Ph.D Candidate, Mathematics
University of California, Santa Barbara; Expected June 2018
Thesis Advisor: David R. Morrison
Thesis Title: On Minimal Models and Canonical Models of Elliptic Fourfolds with Section

M.A., Mathematics
University of California, Santa Barbara; June 2014

B.S., Mathematics, Computer Science minor, Summa Cum Laude
New York University, Tandon School of Engineering; May 2012

Papers

On Minimal Models and Canonical Models of Elliptic Fourfolds with Section

Invited Talks

Towards Minimal Models of Elliptic Fourfolds
Midwest Algebraic Geometry Graduate Conference, UIC ; 2018

Contributed Paper Talks

Towards Minimal Models of Elliptic Fourfolds
Joint Math Meeting, AMS Contributed Paper Session on Algebraic Geometry ; 2018

Posters Presented

Towards Minimal Models of Elliptic Fourfolds
WAGS, UCLA; October 2017

Towards Minimal Models of Elliptic Fourfolds
GAeL XXV, University of Bath; June 2017

Teaching

Teaching Associate (Instructor); University of California, Santa Barbara

Linear Algebra with Applications; Summer 2017

Teaching Assistant; University of California, Santa Barbara

Real Analysis II; Winter 2016
Introduction to Real Analysis; Fall 2016, Fall 2017
Advanced Linear Algebra II; Winter 2015
Advanced Linear Algebra I; Spring 2015, Fall 2015
Transition to Higher Mathematics; Spring 2013, Winter 2017, Spring 2017
Vector Calculus II; Spring 2014
Linear Algebra with Applications; Fall 2012
Calculus II; Winter 2014, Winter 2018
Calculus I; Fall 2013, Fall 2014
Calculus for Social and Life Sciences; Winter 2013

Talks Given

Zariski Decomposition and Beyond
UCSB Graduate Algebra Seminar; 2018

On Minimal Models of Elliptic Fourfold with Section
UCSB Algebraic Geometry Seminar; 2017

Schemes, Un-Sheaf-ed
UCSB Graduate Algebra Seminar; 2017

Fuzzy Sets and Cosets and Groups. Oh My!!
UCSB Graduate Algebra Seminar; 2017

Lifting the Weil: Étale Cohomology
UCSB Graduate Number Theory Seminar; 2017

Zariski Decomposition, and So Can You!
UCSB Graduate Algebra Seminar; 2017

Introduction to the Minimal Model Program: Surfaces I/II/III
UCSB Graduate Algebra Seminar; 2017

Fixed Divisors, If it ain't Broke...
UCSB Graduate Algebra Seminar; 2017

Ghost, Ladders and Permuations
UCSB Graduate Algebra Seminar; 2016

Basic Theory of Elliptic Surfaces
UCSB Algebraic Geometry Seminar; 2016

Riemann-Roch Algebra
UCSB Graduate Algebra Seminar; 2016

Introduction to the Minimal Model Program
UCSB Algebraic Geometry Seminar; 2016

All the Singularities: Put a Ring on it
UCSB Graduate Algebra Seminar; 2016

Belyi's Theorem
UCSB Complex Analysis Reading Seminar; 2015

Blow ups, Blow down, but never Blown over
UCSB Graduate Algebra Seminar; 2015

I scheme, you scheme, we all scheme for...
UCSB Graduate Algebra Seminar; 2015

High School Math to Bezout's Theorem
UCSB Graduate Algebra Seminar; 2014

What Is a Curve in Projective Space?
UCSB Graduate Algebra Seminar; 2014

Fuzzy Sets and Cosets and Groups. Oh My!
UCSB Graduate Algebra Seminar; 2013

Professional Activities

UCSB Graduate Algebra Seminar
Organizer; 2015 - 2018

Conference Attended/To Attend

Midwest Algebraic Geometry Graduate Conference, UIC; 2018
Algebraic Geometry Northeastern Series, Rutgers; 2018
Algebraic Geometry and its Broader Implications, UIC; 2018
Western Algebraic Geometry Symposium, SFSU; 2018
Georgia Algebraic Geometry Symposium, Georgia Tech; 2018
Geometry and Physics of F-theory, Banff, Alberta; 2018
Joint Math Meeting, San Diego; 2018
AMS Sectional Meeting, UC Riverside; 2017
Western Algebraic Geometry Symposium, UCLA; 2017
Géométrie Algébrique en Liberté XXV, University of Bath; 2017
Southern California Algebraic Geometry Seminar, UCSD; 2017
Georgia Algebraic Geometry Symposium, UGA; 2017
Western Algebraic Geometry Symposium, CSU; 2016
Arithmetic Algebraic Geometry, NYU; 2016
Higher Dimensional Algebraic Geometry, University of Utah, Salt Lake City; 2016
Local Global Methods in Algebraic Geometry, UIC; 2016
F-Theory at 20, Caltech; 2016
Southern California Algebraic Geometry Seminar, UCSD; 2016
Southern California Algebraic Geometry Seminar, UCLA; 2015
Joint Math Meeting, Boston; 2012

Abstract

On Minimal Models and Canonical Models of Elliptic Fourfolds with Section

by

David Wen

One of the main research programs in Algebraic Geometry is the classification of varieties. Towards this goal two methodologies arose, the first is classifying varieties up to isomorphism which leads to the study of moduli spaces and the second is classifying varieties up to birational equivalences which leads to the study of birational geometry. Part of the engine of the birational classification is the Minimal Model Program which, given a variety, seeks to find "nice" birational models, which we call minimal models. Towards this direction much progress has been made but there is also much to be done. One aspect of interests is the role of algebraic fiber spaces as the end results of the Minimal Model Program are categorized into Mori fiber spaces, Iitaka fibrations over canonical models and varieties of general type. A natural problem to consider is, starting with an algebraic fiber space, how might it behave with respect to the Minimal Model Program. For case of elliptic threefolds, it was shown by Grassi, that minimal models of elliptic threefolds relate to log minimal models of the base surface. This shows that minimal models, in a sense, have to respect the fiber structure for elliptic threefolds. In this dissertation, I will provide a framework towards a generalization for higher dimensional elliptic fibration and along the way recover the results of Grassi for elliptic fourfolds with section.

Contents

1 Introduction **1**

2 Background **5**

2.1 Precursor to the Minimal Model Program 8

2.1.1 Classical Case of Smooth Surfaces 8

2.1.2 Mori's Program 10

2.2 The Minimal Model Program 13

2.2.1 Minimal Models and Singularities 14

2.2.2 Flips, Flops and Abundance 16

2.2.3 Mori Fiber Spaces, Iitaka Fibrations and Varieties of General Type 18

2.3 Elliptic Fibrations 21

2.3.1 Elliptic Surfaces and Kodaira's Canonical Bundle Formula 22

2.3.2 Generalizations by Fujita, Kawamata and Grassi 24

2.4 Towards Minimal Models of Higher Dimensional Elliptic Fibrations . . . 26

3 Technical Background **27**

3.1 Weierstrass Models 27

3.1.1 Elliptic Fibrations with Section and Weierstrass Equations 28

3.1.2 Singular Elliptic Fibers in Weierstrass Models 30

3.1.3 Weierstrass Models and Collision Points 31

3.2 Higher Dimensional Zariski Decompositions 32

3.2.1 Classical Zariski Decomposition 33
3.2.2 Fujita's Generalization and Birkar's Further Generalization 35
3.2.3 Properties of Fujita-Zariski Decomposition 36
3.2.4 CKM-Zariski Decomposition 37
3.2.5 Birkar's Theorem on Zariski Decompositions and Minimal Models 38

4 Results 40

4.1 Lemmas 40
4.1.1 Fujita-Zariski Decomposition and Resolutions 40
4.1.2 Weierstrass Models and Resolutions 42
4.1.3 Canonical Bundle Formula of a Elliptic Fibration birational to a Weierstrass Model Over a Base with Log Minimal Models 44
4.2 Theorems 51
4.2.1 On Minimal Models of Elliptic Fourfolds with Section 51
4.2.2 On the Canonical Model of Elliptic Fibrations with Section 55
4.2.3 On Minimal Models of Elliptic Fibrations with Section 57

5 Conclusion 58

5.1 Concluding Remarks 58
5.2 Future Direction 59

A Intersection Theory 61

A.1 Smooth Varieties 61
A.1.1 Surfaces 61
A.1.2 Higher Dimensions 63
A.2 Projection Formula 64
A.3 Intersection Theory on Pairs 64

B Related Results **66**

B.1 Simpler Proof of Weaker Statement 66

B.2 Divisors Covered by $K_X + \Delta$-negative curves 68

Chapter 1

Introduction

Algebraic Geometry is one of the oldest fields of mathematics with origins starting with Euclidean geometry of the ancient Greeks, Algebra from the Golden Age of Islam and their unification into coordinate geometry by Decartes in 1637 and independently by Fermat around the same time. This brought about the study of plane curves, where we have curves on the coordinate plane defined by polynomials in two variables.

This idea of studying the algebra of polynomial equations to understand the geometry and vice versa is the seed that grew into the field of Algebraic Geometry. Classical Algebraic Geometry is the study of varieties, i.e. the zero sets of polynomials. In solving polynomial equations, we see the development of complex numbers and their utility as seen in Hilbert's Nullstellensatz that establishes a relation between the algebraic properties of polynomial rings over $\mathbb{C}$ and the geometry properties of the locus of zeros of polynomials in $\mathbb{C}^n$. Further understanding and developments in geometry, led to projective geometry with projective varieties eventually being a natural category for classical algebraic geometry. Some classical theorems are realized in a more elegant way when put into the context of complex projective varieties. For example, Bezout's theorem on the number of intersections of two plane curves on the plane can be fully realized when working on the complex projective plane. These developments solidified classical algebraic geometry as the study of complex projective varieties.

One of the golden idols in mathematics is classification and one of the major research programs that modern algebraic geometers inherited from the classical is the classification of varieties. This is a huge undertaking but two general approaches were developed for such a goal. One is classifying by isomorphism where we identify complex projective varieties up to isomorphism. This leads to the study of Moduli spaces and Geometry Invariant Theory, where the goal is to construct a parameter space that parameterizes isomorphism classes of varieties. The other is a classification by birational maps, which identifies complex projective varieties having isomorphic function fields, where the goal is to classify "nice" varieties within the birational equivalence classes. The engine towards this goal is called the Minimal Model Program (or MMP) which, given any complex projective variety, aims to produces "nice" birational models of which to classify. My research is in the latter, towards understanding and studying the birational classification, and this dissertation will highlight new results in the birational classification of certain classes of complex projective varieties.

The first major results towards a birational classification were achieved by the Italian school of algebraic geometry with the likes of Castelnuovo, Enriques, Noether and more with the birational classification of smooth algebraic surfaces. The classification splits into two steps. The first step is identifying the "nice" birational models, which we call *minimal surfaces*, in the birational equivalence class of a surface and the second step would be classifying these minimal surfaces. The first step was accomplished by Castelnuovo, by blowing down/contracting certain undesirable curves on a surface and showing such a process must terminate. The second was accomplished by Enriques and extended by Kodaira in the Enriques-Kodaira classification of complex surfaces. The success of the algebraic surface case laid the groundwork towards a research program for a higher dimensional analog of a birational classification, where instead of trying to obtain minimal surfaces, we try to obtain *minimal models*.

One potentially glaring gap is the fact that we only considered smooth surfaces in the birational classification but what of singular algebraic surfaces? One of the benefits of algebraic geometry is the ability to understand singular varieties through the algebra of the polynomial rings, so a birational classification of only *smooth* varieties seems to have an inherent gap. This turns out not to be the case due to Hironaka's famed result on resolution of singularities in [12]. In his seminal work, Hironaka showed that in characteristic 0, we can always find a smooth variety in the birational equivalence class of an algebraic variety. In other words, we have that an algebraic variety in characteristic 0 is always birational to a smooth variety. Thus classifying the smooth case will result in a birational classification of all projective varieties. So it is sufficient to classify the smooth case and try to generalize the procedure of the smooth surface case to higher dimensions.

The first breakthrough towards higher dimensions came in 1982 by Mori in [23] using his bend-break lemma to prove his Cone theorem. He showed that starting with a smooth threefold, we can generalize Castelnuovo's results of contracting certain undesirable curves, but a problem arose when it was observed that the result of some of these contraction produced singular varieties. So even working with smooth varieties, this process of contractions can takes us out of the case of smooth varieties. Thus a new approach was needed to get a handle on the possible types of singularities as well as a need to generalize techniques since some were specialized to the smooth case. Also, the introduction of singular varieties into the birational classification resulted in the need for a new type of birational contraction, called a flip, whose existence was unknown and even if it did exist, whether these flips will eventually produce a minimal model is also unknown. By 1988, Mori, with the help of many collaborators, was able the successfully answer these questions for threefolds and showed that any smooth threefold has a minimal model in [22].

The methods of Mori in the threefold case was very constructive, but soon came the realization that applying such a constructive approach to higher dimensions was unfeasible. Thus the methodology changed and a new approach to solving the problem via induction began. With ideas spearheaded by Shokurov and Kawamata, the objects were generalized from varieties to log pairs that lent themselves better to inductive arguments. Thus a generalization was needed to establish the existence of not just *flips* but of *log-flips* in dimension higher than 3. Additionally, there was a need to show that these flips would eventually result in a minimal model. Eventually Kawamata, Matsuda and Matsuki showed the termination of 4-fold (terminal) flips and Shokurov outlined the problems of termination of flips with the ACC conjecture. This eventually culminated in 2010, with the famed paper by Birkar, Cascini, Hacon and Mckernan in [4], where they showed the existence of (klt) flips in all dimensions and the existence of minimal models for varieties of general type in all dimensions through a hefty multi-part induction argument.

Current progress in the minimal model program has extended into two directions. One direction is to try to use the ideas and results of the characteristic zero case and apply it to the finite characteristic to obtain a birational classification of the finite characteristic case. The other direction is towards obtaining and understanding minimal models of varieties not of general type. Towards obtaining minimal models, the termination of arbitrary flips and log flips in higher dimensions is still open. Towards understanding minimal models, some focus has gone towards understanding algebraic fiber spaces since some conjectures imply that fiber spaces play an important role for varieties that are not of general type. Results in this direction relating existence of minimal models of fiber spaces to the base can be seen in [18] and [7]. The results of this dissertation is in this direction towards a generalization of the results in [9] on minimal models of elliptic threefolds to higher dimensions.

Chapter 2

Background

We start with the definition and some properties of a birational map as seen in [11] and [17].

Definition 2.0.1. *Let X, Y be varieties, a rational map from X to Y is an equivalence class of pairs $\langle U, \phi_U \rangle$, where U is an open dense subset of X and $\phi_U : U \to Y$ is a morphism of varieties such that $\langle U, \phi_U \rangle \sim \langle V, \phi_V \rangle$ if and only if $\phi_U|_{U\cap V} = \phi_V|_{U\cap V}$. We frequently denote a rational map as $\phi : X \dashrightarrow Y$ and with gluing we can obtain the maximal open dense $U \subset X$ on which the morphism $\phi : U \to Y$ is defined.*

Definition 2.0.2. *Let X, Y be varieties, a birational map between X and Y is a rational map $\phi : X \dashrightarrow Y$ such that it admits a rational inverse. In other words, we have that there is a $\psi : Y \dashrightarrow X$ such that $\phi \circ \psi = id_Y$ and $\psi \circ \phi = id_X$ as rational maps. If there is a birational map between X and Y, then we say they are birational or birationally equivalent.*

Proposition 2.0.3 ([11, Cor 1.4.5])**.** *For any two k-varieties X and Y, the following are equivalent:*

1. *X and Y are birationally equivalent.*

2. *There exists non-empty open $U \subset X$ and $V \subset Y$, such that U is isomorphic to V.*

3. The field of functions, $K(X)$ and $K(Y)$, are isomorphic as k-algebras.

According to the above proposition, birational geometry is the study of varieties with isomorphic function fields. Compared to isomorphisms of varieties with isomorphic rings of regular functions, birational maps are less rigid so the birational equivalence classes of varieties are bigger with much more flexibility. Often times, the results of birational geometry can be described in two classes, the first is the study of the relations between birationally equivalent varieties and the second is identifying distinguished models in the birational equivalence classes. An example of the former is the Weak factorization of projective varieties as in [25] and an example of the latter is Hironaka's resolution of singularities in [12].

Hironaka's resolution of singularities shows the existence of a smooth birational model in every birational equivalence class of any variety, which is "nice" in the sense that smooth is a very nice property to have. It is this idea of finding useful birational models that motivates the minimal model program, where the goal is to find a birational model of a variety with good algebraic-geometric classification properties. Before proceeding we have the following assumptions and technical details.

A variety will mean an irreducible normal $\mathbb{Q}$-factorial projective variety over $\mathbb{C}$, unless otherwise stated. A **prime divisor** of a variety, X, is a reduced irreducible subvariety of X of codimension 1. A **divisor** is a formal sum of prime divisors, $\sum_i a_i D_i$ where $a_i \in \mathbb{Z}$. A **$\mathbb{Q}$-divisor** is a formal sum of prime divisors where the coefficients can be taken in $\mathbb{Q}$. We say that a $\mathbb{Q}$-divisor $D = \sum_i a_i D_i$ is **effective** if for all i, we have $a_i \geq 0$. A divisor is called **Cartier** if it can be realized locally as the zeros and poles of rational functions coming from a section of the total quotient sheaf. A $\mathbb{Q}$-divisor, D, is call **$\mathbb{Q}$-Cartier** if there is some $0 \neq m \in \mathbb{Z}$ such that mD is Cartier. The property of a variety X being **$\mathbb{Q}$-factorial** is that every divisor is $\mathbb{Q}$-Cartier. Notice that as a part of our definition we do not require our varieties to be smooth some of the properties of X like normal and $\mathbb{Q}$-factorial will give a handle on the singularities of X.

Given a $\mathbb{Q}$-divisor $D = \sum_i a_i D_i$, we define the **round down** $\lfloor D \rfloor = \sum_i \lfloor a_i \rfloor D_i$ and the **round up** $\lceil D \rceil = \sum_i \lceil a_1 \rceil D_i$. A divisor $D = \sum_i a_i D_i$ on a smooth variety, X, is called a **simple normal crossings divisor** if all the prime divisor components D_i intersection transversely, in other words, at each point on D, we have that analytically, it locally looks like a transverse intersection of hyperplanes that do not exceed the dimension of X. We say that two $\mathbb{Q}$-divisors, D_1 and D_2, are **numerically equivalent**, denoted $D_1 \equiv D_2$, if for any curve, $C \subset X$, we have that $D_1 \cdot C = D_2 \cdot C$, where $D \cdot C$ denotes the intersection number (cf. AppendixB) of the divisor D with C. We say that a $\mathbb{Q}$-Cartier divisor D is **nef** if for any curve C, we have that $D \cdot C \geq 0$.

Definition 2.0.4. *A birational morphism from X to Y is a morphism $\phi : X \to Y$ such that there is a non-empty open $U \subset X$ and $V \subset Y$ such that $\phi|_U$ is an isomorphism between U and V. The exceptional set of f, denoted $Ex(f)$, is the set points $x \in X$ such that f^{-1} is not a morphism at $f(x)$.*

Definition 2.0.5. *Let $\phi : X \dashrightarrow Y$ be a birational map and D a prime divisor on X such that $D \cap U \neq \varnothing$ where U is the maximal open subset of X where ϕ is defined, then the birational transform of D, denoted $\phi_* D$, is the closure of the image of $D \cap U$ in Y if $\dim(\phi(D \cap U) = \dim(Y) - 1$, otherwise it will be 0. In other words when $\phi(D \cap U)$ is dense in a divisor of Y we have:*

$$\phi_* D = \overline{\phi(D \cap U)} \subset Y$$

Given a divisors $\Delta = \sum_i a_i D_i$ where D_i are the irreducible prime divisors, we set $\phi_* \Delta = \sum_i a_i \phi_* D_i$. For divisors $D \subset X$ such that $\phi_* D = 0$, we call D an exceptional divisor of ϕ.

Definition 2.0.6. *Let $f : Y \to X$ be a birational morphism with E an exceptional divisor. The center of E over X, denoted $center_X(E)$, is the closure of the image of E.*

Definition 2.0.7. *Given a birational morphism $f : Y \to X$ where mK_Y is a Cartier divisor for $m > 0$, we have the following formula of canonical divisors:*

$$mK_Y = f^*(mK_X) + \sum_i ma_iE_i$$

where E_i are the exceptional divisors of f. We define the discrepancy of E_i over X as $a(E_i, X) := a_i$.

2.1 Precursor to the Minimal Model Program

Following the rough classification of algebraic curves by the genus, the Italian school of algebraic geometry of the 19th century attempted and succeeded in a birational classification of algebraic surfaces. It is the results of the surface case that outlined the process for higher dimensional birational classification, by first obtaining "nice" models which we call minimal models and then eventually classifying these minimal models.

2.1.1 Classical Case of Smooth Surfaces

The classification of algebraic surfaces is accomplished in two parts, the first being the definition and procedure of obtaining these minimal models of smooth surfaces and the second being the classification of these minimal models. The process of obtaining minimal surfaces start with the following theorem of Castelnuovo:

Theorem 2.1.1 (Castelnuovo, cf. [11, Thm. V.5.7])**.** *Let S be a smooth projective surface and $E \cong \mathbb{P}^1$ a curve on S such that $E \cdot E = -1$, then we have that there is a smooth surface S_0 and a map $f : S \to S_0$ such that S is the blow up of a point on S_0 with exceptional divisor E.*

We have that a blow up of a point of a smooth surface results in a surface with an exceptional curve $E \cong \mathbb{P}^1$ with the property that $E \cdot E = -1$, where $E \cdot E$ is the intersection number of E with itself. Castelnuovo's theorem shows that all such curves are a result of a blow up so that we can contract or "blow down" these curves. The following theorem of the base of Neron-Severi ensures that, starting from a smooth surface, one can only blow down a finite number of curves:

Definition 2.1.2 (cf. [19, Prop. 1.1.15])**.** *The Neron-Severi group of a variety X, denoted $NS(X)$, is the group of divisors modulo numerical equivalences.*

Theorem 2.1.3 ([19, Prop. 1.1.16])**.** *The Neron-Severi group is a free abelian group of finite rank.*

By the theorem of the base of Neron-Severi, we have that for smooth varieties, the Neron-Severi group has finite rank. Now, the Neron-Severi group is a quotient of the divisor group by numerical equivalence, thus we will have that each contraction of these exceptional curves with negative self intersection will decrease the rank of the Neron-Severi group from S to S_0 where $f : S \to S_0$ is a blow up of a point. The finite rank of the Neron-Severi group implies that this process of contraction must eventually terminate. Thus, associated to any surface S, we will have that there is a smooth surface $\bar{S}$ such that $f : S \to \bar{S}$ is a sequence of blow ups of points and $\bar{S}$ has no curves of negative self-intersection. We call such a surface $\bar{S}$ a *minimal surface.*

The above shows that starting from a smooth surface we are always able to birationally obtain a minimal surface, so that a classification of minimal surfaces would give a birational classification of smooth surfaces. This was achieved by Enriques for algebraic surfaces and extended by Kodaira to complex analytic surfaces in the Enriques-Kodaira classification of surfaces. We will have that the classification breaks down by Kodaira dimensions a higher dimensional analog of the genus for curves.

Definition 2.1.4. *Let S be a smooth surface, then the Kodaira dimension of S, denoted $\kappa(S)$, is the dimension of the image of $\phi_{|mK_S|}$ where $\phi_{|mK_S|}$ is the rational map induced* *by the linear system $|mK_S|$ for $m \gg 0$. If $|mK_S|$ is not well defined then we say that $\kappa(S) = -\infty$.*

We will have that minimal surfaces, S, with $\kappa := \kappa(S)$ can be classified up to $\kappa = -\infty, 0, 1, 2$. For $\kappa = -\infty$ we will have that S is isomorphic to $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ or is a Del Pezzo surface. For $\kappa = 0$, we will have that S is a $K3$-surface, Enriques surface, abelian surface or bielliptic surface. For $\kappa = 1$, we will have that S is an elliptic surface. For $\kappa = 2$ we will have that S is of general type.

This is the birational classification of surfaces and the goal is to extend these ideas to higher dimension. So given a variety X, we want a process that would birationally modify X (similar to Castelnuovo's Theorem) to a nice birational model $\bar{X}$ where $\bar{X}$ is easier to understand and classify. Then after establishing such a $\bar{X}$, we then proceed to classify them potentially by the Kodaira dimensions. Progress would have to start with the first step and the first significant results in the three dimensional case was by Mori in [23].

2.1.2 Mori's Program

Higher dimensional analogs of Castelnuovo's theorem was quite an undertaking, since the theorem is very specialized towards surfaces. For threefolds, there is no direct translation of curves with negative self intersection and whether it would produce a contraction. Mori's approach in [23] put the problem into a combinatorial setting by looking at the cone of curves and it's intersection pairing with the Neron-Severi group.

Definition 2.1.5 (cf. [17, Def. 1.16]). *Given two 1-cycles, C, C' on X, we say they are numerically equivalent if for any divisor, D, we have that $C \cdot D = C' \cdot D$. We denote the real vector space of 1-cycles modulo numerical equivalence as $N_1(X)$.*

We will have that there is a perfect pairing between $N_1(X)$ and $NS(X) \otimes_{\mathbb{Z}} \mathbb{R}$ and so we have that $N_1(X)$ is a finite dimensional vector space. Furthermore we have the cone of curves:

Definition 2.1.6. *We have that* $NE(X) := \{\sum a_i[C_i] : C_i \subset X, a_i \geq 0\} \subset N_1(X)$ *where* C_i *are irreducible curves and we let* $\overline{NE}(X)$ *to be the closure of* $NE(X)$ *in* $N_1(X)$*. Given a divisor* D *in* X *we define* $\overline{NE}(X)_{D\geq 0}$ *to be the subcone of* $\overline{NE}(X)$ *whose intersection with* D *is* ≥ 0*.*

For a smooth threefold X, instead of contracting curves with negative self intersection, Mori changed the "undesirable curves" to curves whose intersection with the canonical divisor K_X is negative. This is in fact a generalization of the surface case via the adjunction formula. Analyzing the vector space of curves modulo numerical equivalences, Mori showed that there is a cone structure corresponding to the intersection of curves with the canonical divisors with the following theorem:

Theorem 2.1.7 (cf. [17, Thm. 1.24])**.** *Let* X *be a non-singular variety, then:*

- *There are countably many rational curves* $C_i \subset X$ *such that* $0 < -C_i \cdot K_X < \dim(X) + 1$ *and*
$$\overline{NE}(X) = \overline{NE}(X)_{K_X\geq 0} + \sum_i \mathbb{R}_{\geq 0}[C_i]$$
- *For* $\epsilon > 0$ *and ample divisor* H*,*
$$\overline{NE}(X) = \overline{NE}(X)_{K_X+\epsilon H\geq 0} + \sum_{finite} \mathbb{R}_{\geq 0}[C_i]$$

This sets up a nice structure of the cone of curves of a smooth variety X but then Mori showed that extremal rays with curves whose intersections with the canonical divisor is negative produces contractions of X. For the smooth threefold case, Mori classified all such possible contractions as seen in [17, Thm. 1.32] and this leads to a major problems. Some of the contractions results in singularities, which is a departure from the surface case, where minimal surfaces were smooth. It was realized that the models that we are looking for need not be smooth and it is enough for "minimal models" to have **nef** canonical divisors. This realization directed the research into analyzing the possible singularities and one property we require is for our varieties to be $\mathbb{Q}$-factorial, so that we can still have a well defined intersection theory on singular varieties. Additionally, instead of exclusively working in the case of smooth threefold, the class of threefolds expanded to threefolds with at worst terminal singularities (as in definition 2.2.3).

Another problem that arose due to singular varieties was the existence of *small* contractions, where the contraction coming from contracting "extremal" rays of the cone of curve does not contract a divisor but a space of codimension ≥ 2. These are called **small contractions** and are a problem because these contractions result in a variety that is no longer $\mathbb{Q}$-factorial

The solution to this problem was the birational map known as a flip, which topologically is akin to complex codimension 2 surgery.

Definition 2.1.8 (cf. [17, Def. 3.33])**.** *Let X be a normal threefold with a map $f : X \to Y$ a birational morphism such that $Ex(f)$ has codimension ≥ 2 with $-K_X$ being f-ample. Then the flip of $f : X \to Y$ is a birational morphism $f^+ : X^+ \to Y$ with the following:*

1. *K_{X^+} is $\mathbb{Q}$-Cartier.*

2. *K_{X^+} is f^+-ample.*

3. *$Ex(f^+)$ has codimension ≥ 2.*

This implies that all the curves that negatively intersected with K_X that was contracted by f is "flipped" to curves on X^+ whose intersection with K_{X^+} is positive. Quite literally their intersection signs were flipped.

The introduction of this new birational map forced an analysis of these flip maps. The first was to determine whether is was always possible to "flip" a given small contraction arising from the cone theorem and the second after establishing existence we need to show that these flips must eventually terminate to give a minimal model. For contractions of divisors, we have the terminating condition is the rank of the Neron-Severi group but we do not have such a thing for flips. Both of these questions were answered by Mori in [22] for the class of threefolds that contained the smooth threefolds and thus established the existence of minimal models (with at worst terminal singularities) of smooth threefolds.

After establishing the smooth threefold case, the next step would be towards obtaining minimal models of higher dimensional varieties. It turns out some of the techniques of the threefold case becomes computationally infeasible as the dimension of the variety increases. As a result, a different methodology towards obtaining minimal models of higher dimensional minimal models was developed taking Mori's ideas and incorporating cohomology with the intention to use induction to obtain all dimensions. This is the modern version of the problem that we call the Minimal Model Program.

2.2 The Minimal Model Program

The transition towards higher dimensions $\dim(X) \geq 4$, took an approach via cohomology and induction. One generalization that appeared was the use of log pairs (X, Δ) and attempting to run the minimal model program with the log canonical divisor $K_X + \Delta$ instead of just the canonical divisor K_X. So the goal would be to contract $K_X + \Delta$ negative curves. This introduces possibly more singularities but allows for inductive methods that uses techniques of hyperplane section and the adjunction formula. This requires a generalization towards log pairs, where we need to deal with log flips and the possible singularities that may arise.

This brings us back to the situation of flips where we need to show the existence of log flips and showing that sequences of log flips terminate to a minimal model. Mori was able to show the termination of an arbitrary sequence of flips for terminal threefolds, but in higher dimension this is a much harder problem. Thus the approach was to show the termination of classes of flips that would eventually result in a minimal model. It was this approach that was used to show the existence of minimal models of varieties of general type in any dimension in [4] by showing that log flips with scaling terminated.

2.2.1 Minimal Models and Singularities

The methodology of understanding the singularities of the minimal model program uses the notion of resolutions and log resolutions. More specifically we have:

Definition 2.2.1. *A log pair is a pair (X, Δ) where X is a variety and Δ is a divisor of X. Given a log pair (X, Δ) with X normal and $m(K_X + \Delta)$ is Cartier for some $m > 0$ with a birational morphism $f : Y \to X$ then we have the following formula of the log canonical divisor of X:*

$$K_Y = f^*(K_X + \Delta) + \sum_i a_i E_i$$

where E_i are the exceptional divisors of f. We define the discrepancy of E_i over (X, Δ) as $a(E_i, X, \Delta) := a_i$.

Definition 2.2.2. *Let X be a variety, then a resolution of X is a birational morphism $f : Y \to X$ such that Y is a smooth variety. For a log pair (X, Δ), a log resolution is a birational morphism $\phi : Y \to X$ such that Y is smooth and $\phi_*^{-1}\Delta \cup Ex(\phi)$ has simple normal crossing, where $Ex(\phi)$ is the exceptional locus of ϕ in Y.*

It turns out that the values $a(E_i, X)$ and $a(E_i, X, \Delta)$ depend only upon valuations in the field of functions $K(X) \cong K(Y)$ and are not dependent on any choice of f or Y. So that we can take Y to be a log resolution to understand the discrepancy of exceptional divisors. The case of $\Delta = 0$ reduces to the case of discrepancy of exceptional divisors over X which aligns with the classical case of smooth surfaces and terminal threefolds. Additionally, the formula for the canonical divisor for the case of the pair (X, Δ) can actually be reformulated as a sum of exceptional divisors:

$$K_Y + f_*^{-1}\Delta = f^*(K_X + \Delta) + \sum_{E_i \text{ exceptional}} a_i E_i$$

which will be more desirable in later sections. Now with this definition of discrepancy, we can describe the classes of singularities:

Definition 2.2.3 ([17, Def. 2.28, 2.34])**.** *The discrepancy of* (X, Δ) *is given by:*

$$discrep(X, \Delta) := \inf_E \{a(E, X, \Delta) : E \textit{ is an exceptional divisor over } X\}$$

We say that the pair (X, Δ) *is*

- *terminal if* $discrep(X, \Delta) > 0$
- *canonical if* $discrep(X, \Delta) \geq 0$
- *kawamata log terminal (klt) if* $discrep(X, \Delta) > -1$ *and* $\lfloor \Delta \rfloor \leq 0$
- *divisorial log terminal (dlt) if* $discrep(X, \Delta) > -1$ *whenever* $center_X(E) \subset$ *non-snc* (X, Δ)
- *pure log terminal (plt) if* $discrep(X, \Delta) > -1$
- *log canonical (lc) if* $discrep(X, \Delta) \geq -1$

where non-snc (X, Δ)*, denotes the locus of* Δ *that is not of simple normal crossings.*

The case of smooth surfaces and smooth threefold corresponds to the same classes of singularities in their respective their dimensions. We will have that a surface with terminal singularities is in fact smooth and smooth threefolds are in the class of terminal threefold pairs. The different definitions play a role in the minimal model program, we have that running the minimal model program on terminal, canonical, plt and lc log pairs would keep the variety within the same class of singularities. We have that klt pairs is a perfect situation for the induction techniques of higher dimension. With an understanding of the singularities we can define minimal models.

Definition 2.2.4 ([17, Def. 3.50]). *Let (X, Δ) be a log canonical pair and $f : X \to S$ a proper morphism. A pair $(\bar{X}, \bar{\Delta})$ sitting in a diagram:*

$$\begin{tikzcd} X \arrow[rr, dashed, "\phi"] \arrow[dr, "f"'] & & \bar{X} \arrow[dl, "\bar{f}"] \\ & S & \end{tikzcd}$$

is called a minimal model of (X, Δ) over S if:

1. *$\bar{f}$ is proper.*
2. *ϕ^{-1} has no exceptional divisors.*
3. *$\bar{\Delta} = \phi_*(\Delta)$*
4. *$K_{\bar{X}} + \bar{\Delta}$ is $\bar{f}$-nef*
5. *$a(E, X, \Delta) < a(E, \bar{X}, \bar{\Delta})$ for every ϕ-exceptional divisor $E \subset X$.*

2.2.2 Flips, Flops and Abundance

So far we have focused on the properties of minimal models generalizing from the surface and threefold case. But there are still many aspects of the minimal model program and of minimal models that should be highlighted.

Flips

We have that flips are necessary to handle the singularities resulting from small contractions in the minimal model program. It was not until [4], that it was established that flips for *klt* pairs existed in all dimensions. Even so we do not have termination of arbitrary sequences of log flips, what we do know is there there is a class of flips call flips with scaling that can be used to obtain minimal models. Approaches to tackling this problem in general deal with the ACC (Ascending Chain Condition) conjecture, where the log canonical thresholds of varieties of fixed dimension satisfy the ascending chain condition. This gives an upper bound to the number of flips and so would show the termination of flips. Even now there is progress towards termination of log flips of various class of pairs, the most recent being termination of pseudo-effective log flips of log canonical fourfold pairs in [21].

Flops

One aspect of minimal models in higher dimension is the fact that they are not unique, but it can be shown that any two minimal models of a log pair (X, Δ) are isomorphic in codimension 1. In other words we have that two minimal models differ by a codimension ≥ 2 space. This seems very similar to the idea of a flip where we get two varieties that differ by a space of codimension ≥ 2. In the case of minimal models, we call these flops and it turns out that two minimal models are related by a sequence of flops as shown in [13]. A flop is essentially a flip with the exception that the curves involved in the "flop" intersect trivially with the their respective canonical divisors So while we do not have uniqueness we do have a means to relate minimal models of a pair (X, Δ). In this vein, we can ask how many minimal models are their in a birational equivalence class? How many are there up to isomorphism?

Abundance

The abundance conjecture is the means of realizing generalizations of some of the classification properties of the surface case. The conjecture is as stated:

Conjecture 2.2.5 ([17, Conj. 3.12]). *Let (X, Δ) be a log canonical pair with Δ effective,* *then if $K_X + \Delta$ is nef then we have that $|m(K_X + \Delta)|$ is basepoint free for some $m > 0$.*

Why this is useful is that if $|m(K_X + \Delta)|$ is basepoint free then we can get a good understanding of the morphism of a minimal model to it's image via the linear system, which will turn out to be the canonical model (as in definition 2.2.6). Progress in this direction has been mainly centered at algebraic fiber spaces and relating abundance between the base and total space but in general we only know of log abundance for log canonical threefolds with higher dimensions being an open conjecture.

2.2.3 Mori Fiber Spaces, Iitaka Fibrations and Varieties of General Type

After obtaining the minimal model, we would want to some how find a reasonable classification. If the surface case was any indication, we would want first a rough classification by Kodaira dimension which we define via the canonical ring:

Definition 2.2.6. *Let (X, Δ) be a log canonical pair and $f : X \to S$ a proper morphism, then the canonical ring of (X, Δ) is:*

$$R(X, \Delta) := \bigoplus_{m \geq 0} H^0(X, \lfloor m(K_X + \Delta) \rfloor)$$

The canonical model of (X, Δ) over S is defined to be $Proj_S(R(X, \Delta))$. In the case of $(X, 0)$, we suppress the boundary divisor in the notation so we have that $R(X) := R(X, 0)$.

We will have that the canonical ring is a birational invariant. So the canonical model and any other property associated with the canonical ring will also be a birational invariant. Specifically we can define the Kodaira dimension defined as:

Definition 2.2.7. *Let (X, Δ) be a pair, then the Kodaira dimension of (X, Δ), denoted*

$\kappa(X,\Delta)$, is equal to:

$$\kappa(X,\Delta) = \begin{cases} -\infty & : \forall_{m>0}\, H^0(X, \lfloor mK_X + m\Delta \rfloor) = 0 \\ tr.deg(R(X,\Delta)) - 1 & : \textit{Otherwise} \end{cases}$$

where $tr.deg(R(X,\Delta))$ is the transcendence degree of the canonical ring of (X,Δ). Equivalently when $\kappa(X,\Delta) \neq -\infty$, we will have that $tr.deg(R(X,\Delta)$ is equal to the dimension of the canonical model of (X,Δ).

For the case of X being smooth and $\Delta = 0$, the above definition aligns with the classical notion of Kodaira dimension. From the smooth surface case, we grouped the classification of minimal surfaces by their Kodaira dimensions. In higher dimensions, we have a similar rough classification into three classes Mori Fiber Spaces, Varieties of General type and Iitaka fibrations.

Mori Fiber Spaces

Recall that the minimal model programs runs by contracting curves that intersect negatively with $K_X + \Delta$. In this process it is possible to contract curves that cover all of X, for example if we ran MMP on $X = \mathbb{P}^2 \times \mathbb{P}^1$. What results is a map $f : X \to Z$ such that a fiber of f are Fano (having anti-ample canonical divisors). We consider these to be degenerate cases in the Minimal Model program as the contraction is not a birational map. On the other hand this is analogous to the negative Kodaira dimension situation of smooth surfaces since by f we have that X is fibered by varieties of Kodaira dimension $-\infty$. The studies and classification of Mori fiber spaces is known as Sarkisov's program which approaches understanding of Mori fiber spaces through links.

Varieties of General Type

The most well understood varieties of the minimal model program would be varieties of general type. This is due to [4], where we know the existence of minimal models of smooth varieties of general type in all dimensions. Part of the reason why varieties of general type were the first to be understood in all dimensions is due to the fact that the number of minimal models of a pair (X, Δ) is finite and as a result lent itself to an inductive proof. Thus the only thing left is to classify the canonical models of varieties of general type, with a goal of possibly constructing a moduli space of canonical models of varieties of general type.

Iitaka Fibrations

Mori fiber spaces dealt with fibrations of varieties with negative Kodaira dimension and varieties of general type are varieties with top Kodaira dimensions. Iitaka fibrations are a means to address all the intermediate Kodaira dimensions from 0 up $n-1$ where $n = \dim(X)$.

Definition 2.2.8 ([19, cf. Def. 2.1.33])**.** *Let (X, Δ) be pair with X a normal variety and $\kappa(X, \Delta) > 0$. Let ϕ_k be the rational maps determined by the linear series $k(K_X + \Delta)$, then for sufficiently large $k \in \mathbb{N}$ we have that the rational maps $\phi_k : X \dashrightarrow Y_k$ are birationally equivalent to a fixed algebraic fiber space*

$$\phi_\infty : X_\infty \to Y_\infty$$

in the following commutative diagram:

$$\begin{tikzcd} X & X_\infty \arrow[l, "\mu_\infty"'] \arrow[d, "\phi_\infty"] \\ Y_k \arrow[from=u, dashed, "\phi_k"'] & Y_\infty \arrow[l, dashed, "\nu_k"] \end{tikzcd}$$

where μ_∞ and ν_k are birational, $\dim(Y_\infty) = \kappa(X, \Delta)$ *and if we let $\mathcal{L}$ be the line bundle associated with the pullback of $\mathcal{O}_X(m(K_X + \Delta))$ via μ_∞, we will have that $\mathcal{L}$ restricted to a very general fiber, F, of ϕ_∞ has $\kappa(F, \mathcal{L}|_F) = 0$. We call $\phi_\infty : X_\infty \to Y_\infty$ the Iitaka* *fibration associated with* (X, Δ) *and it is unique up to birational equivalence.*

Thus we have that in the classical case, for a variety $X = (X, 0)$ where $\kappa(X) \geq 0$ with at worst terminal singularities, it admits an rational map $X \dashrightarrow Y_m$ coming from the linear series $|mK_X|$, that is birationally equivalent to the Iitaka fibration.

Combining this with the abundance conjecture, if X is a minimal model then for $m \gg 0$ we have that $|mK_X|$ is basepoint free so ϕ_m the map induced by the linear system is in fact a morphism. We will have that ϕ_m is birational to the Iitaka fibration and also a general fiber of ϕ_m will have trivial Kodaira dimension with $\dim(Y_m) = \kappa(X)$. It will also turn out that Y_m is of general type. So to classify the intermediate Kodaira dimension, the Iitaka fibration with abundance implies it is sufficient to classify general type varieties and varieties of Kodaira dimension 0.

Trichotomy

From the above, the goal of classification breaks down to classifying Mori fiber space, varieties of general type and varieties of Kodaira dimension 0. This analog can be seen with the case of surfaces where the classification ends up with fibrations by Fano varieties in the $\kappa = -\infty$ case. We have that for the $\kappa = 0$ case we have $K3$-surfaces and abelian surfaces. We have that surfaces with $\kappa = 1$ are all elliptic surfaces thus are fibered by Kodaira dimension 0 fibers. And lastly $\kappa = 2$ are general type surfaces. This is an extension of the algebraic curve case where it breaks down by genus with $g = 0, 1, \geq 2$, of rational curves, elliptic curves and hyperelliptic curves.

2.3 Elliptic Fibrations

To better understand minimal models and Iitaka fibrations, the property we focus on is that given a rational map $X \dashrightarrow W$ with general fiber of Kodaira dimension 0, it must factor through the Iitaka fibration. With abundance and properties of the canonical model, this hints towards a relation between minimal models of X and W.

To investigate this direction the first non-trivial example of a fiber space with fibers of Kodaira dimension 0 is an elliptic fibration. Using the case of elliptic surfaces as an example, this section outlines the results of elliptic surfaces and threefolds and their minimal models. We start with a definition:

Definition 2.3.1. *An elliptic fibration is a morphism $f : X \to B$ between varieties such that for a general point $x \in B$ we have that $f^{-1}(x)$ is an elliptic curve. We say that $f : X \to B$ an elliptic fibration with section if in addition we have that there is a section $s : B \to X$ such that $f \circ s$ is the identity morphism on B. The ramification locus $\Sigma_{X/B}$ is the set of all points in B such that f is smooth over $B - \Sigma_{X/B}$. We write Σ for $\Sigma_{X/B}$ when the fibration is clearly stated.*

2.3.1 Elliptic Surfaces and Kodaira's Canonical Bundle Formula

The case of elliptic surfaces was extensively investigated by Kodaira in [15] and it resulted in the following theorem that reveals information about minimal models of elliptic surfaces.

Theorem 2.3.2 ([15],[6, Thm 2.9], [9, Thm 0.0])**.** *Let $f : X \to C$ be a relative minimal elliptic fibration with X a surface. Denote by $X_{S_i} = m_i F_i$ the multiple fibers. Then:*

1. *$f_*(K_{X/C})$ is locally free of rank 1*
2. *$\chi(\mathcal{O}_X) = \deg(f_*(K_{X/C})) \geq 0$*
3. *$K_X = f^*(K_S + f_*(K_{X/C})) + \mathcal{O}_X(\sum_i (m_i - 1)F_i)$*
4. *$12 f_*(K_{X/C}) = \mathcal{O}_S(\sum 12 a_k D_k) \otimes \mathbb{J}_\infty$, where $12a_k \in \mathbb{N}$. The number $e^{2\pi i a_k}$ depend on the type of the singular fiber over D_k and are described explicitly in [15]. If we*

write $\mathbb{J}_\infty = \sum b_j B_j$ *then* $\mathbb{J}$ *has a pole of order* b_j *along* B_j. *Thus* $12 f_*(K_{X/S})$ *is a divisor supported on* Σ.

We have that the canonical divisor of X is in fact a pullback of a $\mathbb{Q}$-divisor on C by rewriting the above canonical bundle formula as:

$$K_X = f^* \left(K_S + f_*(K_{X/C}) + \sum_i \frac{m_i - 1}{m_i} F_i \right)$$

This shows that X is not just a relative minimal model over f, but for appropriates base curves C we will have that it will also be a minimal model since any curve of X has non-negative intersection with K_X since it is a pullback of a $\mathbb{Q}$-divisor on C. This implies that for a elliptic surface it is sufficient to run a relative minimal model program to obtain a minimal model of the elliptic surface that is still an elliptic surface. This idea was used by Grassi for the elliptic threefold case in [9] and will be used for the higher dimensional case in the following chapter.

Part of understanding of Kodaira's canonical bundle formula comes of the classification of the singular elliptic fibers of minimal smooth elliptic surfaces. We will have that $f_*(K_{X/S})$ is associated to a divisor on C that supports the singular elliptic fibers. So we have that $12 f_*(K_{X/C}) = \mathcal{O}_S(\sum 12 a_k D_k) \otimes \mathbb{J}_\infty$, where D_k is a divisor that supports the fibers denoted in the table below and a_k corresponds to the singular fibers supported on D_k.

Singular Fiber Type	$a_i = a_i(D_i)$
II	$\frac{1}{6}$
II^*	$\frac{5}{6}$
IV^*	$\frac{4}{6}$
IV	$\frac{2}{6}$
III	$\frac{1}{4}$
III^*	$\frac{3}{4}$
${}_mI_0$	0
I_0^*	$\frac{1}{2}$
${}_mI_b$	0
I_b^*	$\frac{1}{2}$

These values were realized classically by Kodaira via monodromy calculations but it has also been confirmed that the coefficients arise as the log canonical thresholds of the singular fibers. This direction opens up possibilities of understanding properties of singular elliptic fibers in higher dimensional elliptic fibrations.

2.3.2 Generalizations by Fujita, Kawamata and Grassi

Going into higher dimension, there have been generalizations of different parts of Kodaira's theorem above. The theorem doesn't generalize directly but with birational modifications of the base and total space, portions of the theorem can be realized in higher dimensions. Below we have a theorem of Kawamata that gives the same formulation of $f_*\omega_{X/B}$ being associated with divisors that support singular elliptic fibers.

Theorem 2.3.3 ([14], [9, Thm 0.1]). *Let $f : X \to B$ be an elliptic fibrations between smooth varieties. Suppose the ramification divisor is a divisor with simple normal crossings. Then the modular function $\mathbb{J}$ extends to a holomorphic map $B \to \mathbb{P}^1$, $12f_*\omega_{X/B}$ is* *an invertible sheaf and*

$$12f_*\omega_{X/B} \cong \mathcal{O}_B(\sum a_i D_i) \otimes \mathbb{J}_\infty$$

where D_i are irreducible components of Σ and a_i are the rational numbers corresponding to the type of singular fibers over a general points of D_i as in the case of elliptic surfaces.

Part of the assumption is for the ramification locus, Σ, to be a simple normal crossing divisor. This will imply that the $\mathbb{J}$-invariant map is a morphism from $B \to \mathbb{P}^1$. Using these same assumptions, Fujita was able to show that the canonical bundle formula in fact generalizes to elliptic fibrations of any dimension.

Theorem 2.3.4 ([6], [9, Thm 0.2]). *Let $f : X \to B$ be an elliptic fibration between smooth varieties. Assuming that the modular function $\mathbb{J}$ extends to a holomorphic map $B \to \mathbb{P}^1$ then:*

$$\omega_X^{\otimes m} = \pi^* \left(\omega_B^{\otimes m} \otimes \pi_* \omega_{X/B}^{\otimes m} \otimes \mathcal{O}_B \left(m \sum \left(\frac{m_i - 1}{m_i} Y_i \right) \right) \right) \otimes \mathcal{O}_X(mE - mG) \qquad (2.1)$$

where the fiber over the general point of Y_i is a multiple fiber of multiplicity m_i, m is a multiple of $\{m_i\}$, mE and mG are effective disivors. Furthermore $E_{\pi^{-1}(Z)}$ is a union of a finite number of proper transforms of exceptional curves for a generic curve Z and the codimension of $\pi(G)$ is at least 2.

This formula for the elliptic fibrations with a nice "Zariski-type" decomposition of the canonical divisor of elliptic threefolds, allowed Grassi to show the following theorem by running a relative minimal model program and showing the appropriate divisors in the canonical divisor of X is contracted.

Theorem 2.3.5 ([9, Thm 1.1]). *Let $X_0 \to S_0$ be an elliptic threefold which is not unirruled. Then there exists a birationally equivalent fibrations $\bar{\pi} : \bar{X} \to \bar{S}$, such that $\bar{X}$ has at worst terminal and $\bar{S}$ log terminal singularities. Furthermore $K_{\bar{X}}$ is nef and*

$K_{\bar{X}} \equiv \bar{\pi}^*(K_{\bar{S}} + \bar{\Lambda})$, *where* $\bar{\Lambda}$ *is a* $\mathbb{Q}$*-boundary divisor. Thus the canonical bundle is a pullback of a* $\mathbb{Q}$*-bundle on* $\bar{S}$*.*

So this realizes the minimal model aspect of Kodaira's theorem, where running a relative version of the minimal model program is sufficient to obtain a minimal model of an elliptic threefold that is still an elliptic threefold whose canonical divisor is numerically equivalent to a pullback of a $\mathbb{Q}$-divisor on the base (which turns out to be a log minimal model).

2.4 Towards Minimal Models of Higher Dimensional Elliptic Fibrations

The results above were developed in the late 80's and early 90's with a further result on obtaining equidimensional minimal models of elliptic threefolds with section in [8]. Advancing towards higher dimensions would need further development of the minimal model program. Grassi was able to run the relative minimal model program due to the fact that Mori showed the minimal model program works for smooth threefolds. At the time theorems towards the fourfold case and higher dimensions were not as developed. Of course eventually, the smooth fourfold case was realized and so the possibility of realizing Grassi's theorem in higher dimension was possible, but complications in higher dimensional birational geometry needed to be addressed. For example, in higher dimensions it is possible we have a flip on the base, it is unknown what happens to the fibration and to singular fibers over the flip locus.

Additionally, we do not have a nice "Zariski-type" decomposition of the canonical divisor of elliptic fibrations of dimension higher than 3. In fact it can be shown that unlike the classical Zariski decomposition on surfaces, Zariski decomposition in higher dimensional varieties need not even exist! Thus this aspect needs to be addressed if we are intending to generalize Grassi's results.

Chapter 3

Technical Background

The following chapter is meant to address the problems that arise with generalizing Grassi's theorem to higher dimensions. The first section will introduce Weierstrass models which will give local equations to that describes elliptic fibrations with section. This helps with understanding singular fibers over the base and helps towards addressing the problems of understanding the fibration due to flips on the base. The second section introduces Zariski decomposition and higher dimensional generalizations to get a handle on the canonical divisor of the elliptic fibrations as seen in [6].

3.1 Weierstrass Models

In the 1 dimensional case, we have that elliptic curves over $\mathbb{C}$ can be represented by a Weierstrass equation of the form:

$$y^2 = x^3 + fx + g$$

where $f, g \in \mathbb{C}$ and the discriminant, $4f^3 + 27g^2$, is non-zero. From the definition of elliptic fibration in 2.3.1, it is not clear if it is possible to recover an analog of a Weierstrass equation for elliptic fibration, which we call a *Weierstrass model*. It turns out that it is not always possible to obtain a such a model since it is associated with the existence of a point on the genus 1 curve. So to obtain this Weierstrass model, we need the existence of a section, hence the definition of an elliptic fibration with section.

This portion is a review of the relation between properties of elliptic fibrations with section and Weierstrass models. The first portion deals with the birational relations between an elliptic fibration with section and Weierstrass models. The second section deals with understanding the singular fibers of elliptic fibrations, from Kodaira's classification of singular fibers, through Weierstrass Models. The last portion deals with higher dimensional issues arising due to the "collision" of singular fibers in elliptic fibrations.

3.1.1 Elliptic Fibrations with Section and Weierstrass Equations

This is a review of the construction of a Weierstrass Model and it's birational relation to elliptic fibrations with section from [24]. This implies that birationally, working with elliptic fibrations with section is equivalent to working with Weierstrass models.

Let S be a complex variety and $\mathcal{L}$ a line bundle with f and g sections of $\mathcal{L}^{-4}$ and $\mathcal{L}^{-6}$ such that $\Delta := 4f^3 + 27g^2$, a section of $\mathcal{L}^{-12}$ is not identically zero on S. Let $\mathbb{P} := P_S(\mathcal{O}_S \oplus \mathcal{L}^2 \oplus \mathcal{L}^3)$, be the projectivization of the vector bundle $\mathcal{O}_S \oplus \mathcal{L}^2 \oplus \mathcal{L}^3$ over S. This gives a projection map $p : \mathbb{P} \to S$, such that over each point $s \in S$, we have that $p^{-1}(S)$ is isomorphic to $\mathbb{P}^2$.

Let X, Y and Z be sections of $\mathcal{O}_{\mathbb{P}}(1) \otimes \mathcal{L}^{-2}$, $\mathcal{O}_{\mathbb{P}}(1) \otimes \mathcal{L}^{-3}$ and $\mathcal{O}_{\mathbb{P}}(1)$, where $\mathcal{O}_{\mathbb{P}}(1)$ is a tautological bundle of $\mathbb{P}$ and these sections, respectively, correspond to:

$$\mathcal{L}^2 \longrightarrow \mathcal{O}_S \oplus \mathcal{L}^2 \oplus \mathcal{L}^3$$
$$\mathcal{L}^3 \longrightarrow \mathcal{O}_S \oplus \mathcal{L}^2 \oplus \mathcal{L}^3$$
$$\mathcal{O}_S \longrightarrow \mathcal{O}_S \oplus \mathcal{L}^2 \oplus \mathcal{L}^3$$

We will have that X, Y, Z behave as global coordinates of $\mathbb{P}$ and allows for defining a Weierstrass model as follows.

Definition 3.1.1. *Using the above notation, we denote $W(\mathcal{L}, f, g)$ as the divisor in $\mathbb{P}$ defined by the equation, $Y^2Z - (X^3 + fXZ^2 + gZ^3)$, and we call this a Weierstrass model over S.*

We have the following properties for $\pi : W := W(\mathcal{L}, f, g) \to S$:

1. W is a complex variety and π is a proper flat surjective morphism with fibers being irreducible cubic curves in $\mathbb{P}^2$.

2. If S is normal then W is normal.

3. The projection $\mathcal{O}_S \oplus \mathcal{L}^2 \oplus \mathcal{L}^3 \to \mathcal{L}^3$ gives a section $s : S \to \mathbb{P}$ such that $\sigma(S) \subset W$ such that $\sigma(S)$ is a Cartier divisor on W. We call $\sigma(S)$ the canonical section and denote it by $\Sigma(\mathcal{L}, f, g)$. We have that $\Sigma(\mathcal{L}, f, g)$ behaves like the point at infinity for elliptic curves in $\mathbb{P}^2$.

4. Locally over S we have that W is locally a Weierstrass equation, in the sense that given an open affine $U = Spec(A) \subset S$, we have an open affine $V \subset \pi^{-1}(U)$ is isomorphic to:

$$y^2 = x^3 + fx + g$$

where $f, g \in A$ and $4f^3 + 27g^2 \neq 0 \in A$.

These properties establish that a Weierstrass model is an elliptic fibration with section and give us local equations to work with in understanding the fibration. This also establishes that not all elliptic fibrations with section are Weierstrass models since the definition of elliptic fibration with section allows for fibers that are not equidimensional which would violate flatness of a Weierstrass model. But we have via Nakayama the following theorem establishing the birational connection between elliptic fibrations with section and Weierstrass models.

Theorem 3.1.2 ([24, Theorem 2.1])**.** *Let $\pi : X \to S$ be an elliptic fibration with section between complex manifolds. Then there exists a Weierstrass model $W(\mathcal{L}, f, g)$ over S and a proper birational morphism $\mu : X \to W(\mathcal{L}, f, g)$ over S such that $\sigma(S) = \mu^*(\Sigma(\mathcal{L}, f, g))$.*

Thus understanding the birational properties of a Weierstrass model gives birational properties of elliptic fibrations with section. This is useful is because Weierstrass models are defined by local equations that we know and this will give a better handle of singular elliptic fibers.

3.1.2 Singular Elliptic Fibers in Weierstrass Models

The singular elliptic fibers of Weierstrass models are easy to understand by knowing the order of vanishing of the base from [20]. Given a Weierstrass model, we have that locally it is of the form:

$$y^2 = x^3 + fx + g$$

where f, g are local equations on a base S. We have that the singular fiber over $x \in S$ can be determined modulo $(4, 6, 12)$ by (L, K, N) where L is the order of vanishing of f at x, K is the order of vanishing of g at x and N is the order of vanishing of the discriminant, $4f^3 + 27g^2$ at x. It turns out that for Weierstrass models we have the following table of singular fibers by the following order of vanishing.

Singular Fiber Type	$a_i = a_i(D_i)$	(L, K, N)
II	$\frac{1}{6}$	$(L \geq 1, 1, 2)$
II^*	$\frac{5}{6}$	$(L \geq 4, 5, 10)$
IV^*	$\frac{4}{6}$	$(L \geq 2, 2, 4)$
IV	$\frac{2}{6}$	$(L \geq 3, 4, 8)$
III	$\frac{1}{4}$	$(1, K \geq 1, 3)$
III^*	$\frac{3}{4}$	$(3, K \geq 5, 9)$
I_0	0	$(L, 0, 0)$
I_0	0	$(0, K, 0)$
I_0^*	$\frac{1}{2}$	$(L \geq 2, K \geq 3, 6)$
I_b	0	$(0, 0, N)$
I_b^*	$\frac{1}{2}$	$(2, 3, N \geq 7)$

A slight difference between the above table and Kodaira's classification of singular elliptic fibers is that there are no multiple fibers ${}_mI_b$ and ${}_mI_b^*$ since the existence of a section implies that all the fibers have multiplicity 1. So the existence of a section makes things less general but allows for simpler calculations for understanding of the properties of the fibration structure, especially in higher dimensions. This is apparent in understanding the interactions between singular fibers in higher dimensions.

3.1.3 Weierstrass Models and Collision Points

The above understanding of singular fibers of Weierstrass models allows us to understand the so called "collision" points. A collision point is where we have intersection of singular fibers of elliptic fibrations. For example, consider the Weierstrass model over $\mathbb{C}^2$:

$$y^2 = x^3 + stx + st$$

where $(s,t) \in \mathbb{C}^2$. According to the table above, we will have that for a general point on the divisors $s = 0$ and $t = 0$, we have that it supports a type II singular fiber. But we have that at $(0,0) \in \mathbb{C}^2$, the intersection of $s = 0$ and $t = 0$, we have the singular fiber is a "collision" of two type II singular fibers. From the table we see that it is a type IV fiber over $(0,0)$.

There is no general approach towards understanding collision points for elliptic fibrations but working with Weierstrass model allows us to understand the singular fibers and collision points via orders of vanishing on the base. This approach and analysis of collision points of Weierstrass threefolds was done by [20] to obtain flat resolutions of Weierstrass threefolds and studied further in [8].

In higher dimensions, this allows an understanding of the behavior of the total space after a birational transformation of the base of a Weierstrass model. Blowing up the base of a Weierstrass model, we can pull the local equation back to obtain a Weierstrass model over the new base and this analysis allows us to understand the resulting singular fibers of the new Weierstrass model.

3.2 Higher Dimensional Zariski Decompositions

The Zariski Decomposition was first developed by Oscar Zariski in [26] for studying the plurigenera of effective divisors on surfaces. Further generalizations by Fujita extended the results to pseudoeffective divisors and it's applications is closely linked to the minimal model program for surfaces. This section will begin with a review of the classical surface case of Zariski decomposition and its properties, followed by more modern results on higher dimensional generalizations and its relationship with the higher dimensional minimal model program.

3.2.1 Classical Zariski Decomposition

Theorem 3.2.1 ([19, Thm 2.3.19], [1, Thm 14.14])**.** *Let X be a smooth projective surface and D a pseudoeffective integral divisor on X. Then we have that D can be uniquely written as:*

$$D = P + N$$

where P, N are $\mathbb{Q}$-divisors with the following properties:

- *P is nef*
- *$N = \sum_{i=1}^{n} a_i N_i$ is effective and if $N \neq 0$, we have that the intersection matrix $(N_i \cdot N_j)$ is negative definite.*
- *For all i, we have that $P \cdot N_i = 0$*

We have that the Zariski decomposition have the following properties:

Proposition 3.2.2 ([1, Lemma 14.17], cf. [19, Prop. 2.3.21])**.** *Given a Zariski decomposition $D = P + N$ on a nonsigular projective surface X, we have that the natural map:*

$$H^0(X, mP) \longrightarrow H^0(X, mD)$$

is an isomorphism for all $m \in \mathbb{N}$.

Thus the plurigenera of mD behaves like the plurigenera of the nef divisor mP. So this is a cohomological condition due to the Zariski decomposition of a divisor D, the next property is a birational property of a Zariski decomposition.

Proposition 3.2.3. *Given a Zariski decomposition $D = P + N$ on a nonsigular projective surface X and $f : W \to X$ a birational morphism from a smooth surface W, we have that if $f^*(D) = P' + N'$ where P' is nef and N' is effective then we have that $P' \leq f^*P$.*

Proof. We have that $f^*(D)$ is pseudoeffective since D is pseudoeffective. This implies that $f^*(D)$ has a Fujita-Zariski decomposition,which will be $f^*(P) + f^*(N)$. To show this we reduce to the case where f is a single blow up, we can do this since f is a birational morphism of smooth surfaces and so it is sequence of blowups and if a blow up preserves the Zariski decomposition then certainly a sequence of blow ups will.

Now assume that f is the blow up of a point. Then we have that $f^*(D)$ intersects trivially with the exceptional curve. If E does not appear in the support of $f^*(N)$, then we are done since this implies the intersection matrix of the curves in the support of $f^*(N)$ is the same as the intersection matrix of the curves in the support of N. If E appears in the support of $f^*(N)$ then by [1, Lemma 14.12], we have that the support of $f^*(N)$ are the same as the curves in the support of N with the addition of E with $E \cdot f^*(D) = 0$, and so we have that the intersection matrix of the curves in the support of $f^*(N)$ is negative definite. Additionally, we have that $E \cdot f^*(P) = 0$ which paired with the fact that intersections are preserved by pullback we have that $f^*(D) = f^*(P) + f^*(N)$ is a Zariski decomposition of $f^*(D)$ and is unique.

Now we have that $f^*(D) = f^*(P) + f^*(N) = P' + N'$. Rearranging the terms we have that $f^*(P) - P' = N' - f^*(N)$ and we want to show that $N' - f^*(N)$ is effective. Now since $f^*(D)$ is the positive part of a Zariski decomposition and P' is nef, we have that for any curve in the support of $f^*(N)$ we get $(f^*(P) - P') \cdot C \leq 0$. By [1, Lemma 14.15], we have that this implies that $N' - f^*(N)$ is effective and so $f^*(P) - P'$ is effective. So we do get $P' \leq f^*(P)$. ■

The birational properties of the Zariski decomposition play a significant role in understanding the minimal model program for surfaces mainly by placing $D = K_X$ where K_X is the canonical divisor of X. It will turn out that the Zariski decomposition of $K_X = P + N$, lays a guides for obtaining a minimal model. Specifically, we have that P is going to be the pullback of the canonical divisor of a "minimal" model and N is the negative effective part consisting of curves to be contracted.

Applications of the Zariski decomposition for surfaces are well documented and some of modern day research is devoted to finding a general Zariski decomposition for higher dimensional varieties. Unfortunately, there is no easy solution towards generalization since often times it does not encompass the full power of the original Zariski decomposition and so the trend has been to impose the sought after property into the definition. This leads to two type of generalized Zariski decomposition for this dissertation, the Fujita-Zariski decomposition and the CKM-Zariski decomposition.

3.2.2 Fujita's Generalization and Birkar's Further Generalization

The Fujita-Zariski decomposition was first introduced in [6] as a means to understand and study the canonical rings of elliptic threefolds. This legacy makes the Fujita-Zariski decomposition a reasonable and useful tool in understanding elliptic fibrations in general and plays a pivotal role in approaching the results of this disseration.

Definition 3.2.4 ([6, Def 1.18]). *A $\mathbb{Q}$-divisor D on a (manifold) M admits a Fujita-Zariski Decomposition if there exists a birational morphism $\pi : M' \to M$ and an effective $\mathbb{Q}$-divisor N on M' such that N is numerically fixed by π^*D and $P = \pi^*D - N$ is nef.*

Definition 3.2.5 ([6, Def 1.7]). *An effective $\mathbb{Q}$-divisor E on M is said to be numerically fixed by a $\mathbb{Q}$-divisor D if for any birational morphism $\pi : X \to M$, we have that $\pi^*(E)$ clutches $\pi^*(D)$.*

Definition 3.2.6 ([6, Def 1.7]). *An effective $\mathbb{Q}$-divisor E on M is said to to clutch a $\mathbb{Q}$-divisor D if $F - E$ is effective, for any effective $\mathbb{Q}$-divisor F where $D - F$ is nef.*

This was the original definition give by Fujita but recently there has been a more modern definition embedded with the birational property of the classical Zariski decomposition.

Definition 3.2.7 ([3, Def 1.1]). *Let D be an $\mathbb{R}$-Cartier divisor on X, a normal variety. A Fujita-Zariski Decomposition of D is an expression $D = P + N$ such that:*

- *P and N are $\mathbb{R}$-Cartier*
- *P is nef and $N \geq 0$*
- *If $f : W \to X$ is a projective birational morphism from a normal variety and $f^*(D) = P' + N'$ with P' nef and N' effective, then $P' \leq f^*(P)$.*

Proposition 3.2.8. *Let D be a $\mathbb{Q}$-Cartier divisor on a smooth projective variety X. Then the two definitions of Fujita-Zariski decompositions of D are equivalent.*

Proof. Let $D = P + N$ be a Fujita-Zariski decomposition in the sense of Fujita [6]. We will show that this implies Birkar's definition of Fujita-Zariski decomposition. Let $f : X' \to X$ be a birational morphism with $f^*(D) = P' + N'$ where P' is nef and N' is an effective $\mathbb{Q}$-Cartier divisor. We have that N is numerically fixed by D and so $f^*(N)$ clutches $f^*(D)$. Thus since $f^*(D) - N' = P'$ is nef we have that $N' - f^*(N)$ is effective. But we know that $N' = f^*(D) - N$ and $f^*(N) = f^*(D) - f^*(P)$. So we have that replacing and simplifying we have that $f^*(P) - P'$ is effective.

Let $D = P + N$ be a Fujita-Zariski decomposition in the sense of Birkar [3] and $f : X' \to X$ a birational morphism. We will show that N is numerically fixed by D, so we wish to show that $f^*(N)$ clutches $f^*(D)$. Let N' be an effective $\mathbb{Q}$-divisor such that $f^*(D) - N' = P'$ is nef, then we want to show that $N' - f^*(N)$ is effective. We know that since $P + N$ is a Fujita-Zariski decomposition in the Birkar sense, we have that $f^*(P) - P'$ is effective. Replacing with $f^*(P) = f^*(D) - f^*(N)$ and $P' = f^*(D) - N'$, we get that $N' - f^*(N)$ is effective. Thus showing that the two definitions are equivalent.

■

3.2.3 Properties of Fujita-Zariski Decomposition

Below we have a sequence of propositions of properties of a Fujita-Zariski decomposition with respect to surjective morphism between complex manifolds that will be useful for later discussions.

Proposition 3.2.9 ([6, Prop. 1.8]). *Let E be an effective $\mathbb{Q}$-divisor on M and suppose that E is numerically fixed by a Cartier divisor L on M. Then $\bar{E}$ is contained in the fixed by $|L|$, where $\bar{E}$ is the smallest effective Cartier divisor such that $\bar{E} - E$ is effective.*

Proposition 3.2.10 ([6, Prop. 1.10]). *Let $f : M \to S$ be a surjective morphism of manifolds such that any general fiber is connected. Let X be an effective $\mathbb{Q}$-divisor on M such that $\dim(f(X)) < \dim(S)$. Suppose that, for every irreducible component Z of $f(X)$ with $\dim(Z) = \dim(S) - 1$, there is a prime divisor D on M such that $f(D) = Z$ and $D \not\subset$ Supp(X). Then X is numerically fixed by $f^*L + X$ for any $\mathbb{Q}$-divisor L on S.*

Proposition 3.2.11 ([6, Prop. 1.11]). *Let $f : M \to S$ be a surjective morphism of manifolds and suppose that an effective $\mathbb{Q}$-divisor E on S is numerically fixed by a $\mathbb{Q}$-bundle L on S. Then f^*E is numerically fixed by f^*L.*

Proposition 3.2.12 ([6, Prop. 1.22]). *Suppose that an effective $\mathbb{Q}$-divisor E is numerically fixed by a $\mathbb{Q}$-divisor L. Then $L - E$ admits a Fujita-Zariski decomposition if and only if so does L. Moreover the semipositive parts of them are the same.*

Proposition 3.2.13 ([6, Prop. 1.23]). *Let $L = N + H$ be a Fujita-Zariski decomposition on M of a $\mathbb{Q}$-divisor L. Then for any effective $\mathbb{Q}$-divisor F on M such that Supp$(F) \subset$ Supp(N), F is numerically fixed by $F + H$. So that $F + H$ admits a Fujita-Zariski decomposition.*

3.2.4 CKM-Zariski Decomposition

The following is another generalization of the Zariski decomposition but with the cohomological properties embedded into the definition. This is quite useful since, this addresses the plurigenera of a divisor, which plays a role towards the classification of minimal models into Mori Fiber Spaces, Iitaka Fibrations and General Type Varieties.

Definition 3.2.14 ([3, Def 1.2]). *Let D be an $\mathbb{R}$-Cartier divisor on X/Z, a normal variety. A Cutkosky-Kawamata-Morikawa-Zariski Decomposition (CKM-Zariski Decomposition) over Z of D is an expression $D = P + N$ such that:*

- *P and N are $\mathbb{R}$-Cartier*
- *P is nef and $N \geq 0$*
- *The morphism $\pi_* \mathcal{O}_X(\lfloor mP \rfloor) \to \pi_* \mathcal{O}_X(\lfloor mD \rfloor)$ are isomorphisms for all $m \in \mathbb{N}$, where π is the morphism from $X \to Z$.*

While the definitions are somewhat different, we do have a relation. Specifically from [6, Cor. 1.9] and [3], we have that a Fujita-Zariski decomposition is a CKM-Zariski decomposition, so that a Fujita-Zariski decomposition of a divisor D has the cohomological property on the plurigenera of D.

3.2.5 Birkar's Theorem on Zariski Decompositions and Minimal Models

Letting D be a canonical divisor, we can recover a relation of higher dimensional minimal models and generalized Zariski decompositions. It shouldn't be surprising due to applications of various Zariski decomposition in understanding higher dimensional minimal model program but we have the following result of Birkar that solidifies this relation.

Theorem 3.2.15 ([3, Thm 1.5]). *Assume the log minimal model program for $\mathbb{Q}$-factorial divisorial log terminal pairs in dimension $n-1$. Let (X, Δ) be log canonical of dimension n, then the following are equivalent:*

- *$K_X + \Delta$ birationally has a Fujita-Zariski Decomposition*
- *$K_X + \Delta$ birationally has a CKM-Zariski Decomposition*
- *$K_X + \Delta$ birationally has a Weak Zariski Decomposition*

- (X, Δ) *has a log minimal model*

We say that D on X birationally has a Zariski decomposition, if there is a birational morphism $g : W \to X$ such that $g^*(D)$ has a Zariski decomposition. This idea plays a significant role in the results as a means to recapture parts of Fujita's results in [6] of a Fujita-Zariski decomposition for canonical bundles of elliptic threefolds. Then applying a generalized version of Grassi's argument in [9] to elliptic fourfolds with section, we should be able to realize the higher dimensional version of Grassi's theorem.

Chapter 4

Results

4.1 Lemmas

We start with three lemmas that set up the argument for a generalized version of Grassi's results for elliptic threefold. The goal of these lemmas is to obtain a birationally equivalent elliptic fourfold whose canonical divisor admits a Fujita-Zariski decomposition.

4.1.1 Fujita-Zariski Decomposition and Resolutions

This lemma establishes a relation between the Fujita-Zariski decomposition of divisors and birational transformations. More specifically, it says the Fujita-Zariski decomposition is preserved up to a difference of effective exceptional divisors.

Lemma 4.1.1. *Let (B, Δ) be a log pair with B smooth, and $\Delta = \sum d_i D_i$ a simple normal crossing divisor with $\lfloor \Delta \rfloor = 0$ and $(\bar{B}, \bar{\Delta})$ a log minimal model of (B, Δ) with a common log resolution, $\tilde{B}$ with morphisms $g : \tilde{B} \to B$ and $h : \tilde{B} \to \bar{B}$. Let $\tilde{\Delta}$ be a boundary divisor on $\tilde{B}$ such that $K_{\tilde{B}} + \tilde{\Delta} - g^*(K_B + \Delta)$ is an effective exceptional divisor over g, then we have that $K_{\tilde{B}} + \tilde{\Delta}$ has a Fujita-Zariski Decomposition.*

Proof. From the assumptions we have that:

$$K_{\tilde{B}} + \tilde{\Delta} = g^*(K_B + \Delta) + N$$

where N is a g-exceptional effective divisor. From [3], we have that $g^*(K_B + \Delta)$ has a Fujita-Zariski Decomposition, explicitly it is:

$$g^*(K_B + \Delta) = h^*(K_{\bar{B}} + \bar{\Delta}) + E$$

Where E is an h-exceptional effective divisor. Combining the two equations we get:

$$K_{\tilde{B}} + \tilde{\Delta} = h^*(K_{\bar{B}} + \bar{\Delta}) + E + N$$

which we will show is the Fujita-Zariski decomposition of $K_{\tilde{B}} + \tilde{\Delta}$.

We apply proposition 3.2.10 to $g : \tilde{B} \to B$ with respect to N. To verify that we can do so, we have that g is a birational morphism so that a general fiber of g certainly connected. We have that N is a g-exceptional divisors so that its image via g is of a smaller dimension than N. This gives the following inequality $\dim(g(N)) \leq \dim(N) - 1 < \dim(B)$. Thus there is no irreducible component of $g(N)$ that has dimension equal $\dim(B) - 1$ since g contracts N and so $\operatorname{codim}(g(N)) \geq 2$. This satisfies the conditions of the proposition, thus we have that N is numerically fixed by $g^*L + N$ for any $\mathbb{Q}$-divisor L on B.

Let $L = K_B + \Delta$ and we will then have that N is numerically fixed by:

$$K_{\tilde{B}} + \tilde{\Delta} = g^*(K_B + \Delta) + N$$

Now from proposition 3.2.12, since $g^*(K_B + \Delta)$ admits a Fujita-Zariski decomposition, this implies that $g^*(K_B + \Delta) + N$ admits a Fujita-Zariski decomposition and their nef parts are the same. From above we see the nef part of $g^*(K_B + \Delta)$ is $h^*(K_{\bar{B}} + \bar{\Delta})$ and so we have

$$K_{\tilde{B}} + \tilde{\Delta} = h^*(K_{\bar{B}} + \bar{\Delta}) + E + N$$

is the Fujita-Zariski decomposition of $K_{\tilde{B}} + \tilde{\Delta}$. ■

4.1.2 Weierstrass Models and Resolutions

In addition to understanding the behavior of the Fujita-Zariski decomposition with respect to resolutions, we also need to establish an understanding of the relations between resolutions of the base and the Weierstrass models. The lemma below, establishes a constraint on the pullback of the Weierstrass models between different resolutions with exceptional divisors that agree on valuations.

Lemma 4.1.2. *Let $\pi : X \to B$ be a Weierstrass model with the $\mathbb{J}$-invariant map, $B \to \mathbb{P}^1$, being a morphism. For $i = 1, 2$, let $f_i : B_i \to B$ be any log resolutions of (B, Δ), where Δ supports the singular fibers of π with coefficients as determined in Kodaira classification, with E_1 and E_2 being divisors on B_1 and B_2 associated to the same valuation in the function field of B. Let $\pi_i : X_i \to B_i$ be the Weierstrass models obtained by base change, then we have that there exists $U_i \subset E_i$ that is dense in E_i such that $\pi_1^{-1}(U_1) \cong \pi_2^{-1}(U_2)$.*

The statement of the results is a bit technical but it can be summed up as saying that there are sets $U_i \subset E_i$ such that $E_i = \overline{U_i}$ with an isomorphism $\phi : U_1 \to U_2$ such that the fiber over $x \in U_1$ of π_1 is isomorphic to the fiber over $\phi(x) \in U_2$ of π_2. Even shorter, we say that fibers over general points of E_1 correspond to fibers over general points of E_2.

Proof. To start this proof we have the following commutative diagram:

$$\begin{tikzcd}
X_1 \arrow[d, "\pi_1"'] & & X_2 \arrow[d, "\pi_2"] \\
B_1 \arrow[rr, dashed, "\phi"] \arrow[dr, "f_1"'] & & B_2 \arrow[dl, "f_2"] \\
& (B, \Delta) &
\end{tikzcd}$$

where ϕ is a birational map and X_1 and X_2 obtained by base change so that $X_i = X \times_B B_i$, so we have that X_1 and X_2 are birational equivalent Weierstrass models over birationally equivalent bases. Now consider a common log resolution of B_1 and B_2 and further base change of the Weierstrass models, so we have the following:

$$\begin{tikzcd}
& X_3 \arrow[dl, "\psi_1"'] \arrow[dr, "\psi_2"] \arrow[dd, "\pi_3"] & \\
X_1 \arrow[ddd, "\pi_1"'] & & X_2 \arrow[ddd, "\pi_2"] \\
& B_3 \arrow[ddl, "g_1"'] \arrow[ddr, "g_2"] & \\
& & \\
B_1 \arrow[rr, dashed, "\phi"] \arrow[dr, "f_1"'] & & B_2 \arrow[dl, "f_2"] \\
& (B, \Delta) &
\end{tikzcd}$$

Now since E_1 and E_2 have the same valuation of the function field, we have that on B_3 their strict transform by g_1 and g_2 respectively agree and are a divisor E on B_3. Since E is not contracted by g_i we have that there is a set $U \subset E$ dense in E such that g_i is an isomorphism on U. Thus we have that there are sets $U_i \subset E_i$ dense in E_i such that $U_1 \to U \to U_2$ are isomorphisms. Now we analyze the fibers over E and E_i.

We have that the Weierstrass model $\pi_3 : X_3 \to B_3$ is obtained by taking the base change and since composition of base changes is a base change we have the following for $i = 1, 2$:

$$X_i \times_{B_i} B_3 = (X \times_B B_i) \times_{B_i} B_3 = X \times_B B_3$$

Thus the pullbacks of the Weierstrass models are isomorphic. Now since $\pi^{-1}(E)$ is not exceptional with respect to ψ_i, we have that over $U \subset E$, ψ_i is an isomorphism of $\pi_3^{-1}(U)$ onto is image. So we have that there are isomorphisms $\pi_1^{-1}(U_1) \to \pi_3^{-1}(U) \to \pi_2^{-1}(U_2)$. ■

This lemma is important for the next lemma because this states that a general fiber over the exceptional divisor of a Weierstrass model is invariant over general points. So that given exceptional divisors that correspond to the same valuation in the function field, the general fibers of the Weierstrass model over these divisors are isomorphic.

4.1.3 Canonical Bundle Formula of a Elliptic Fibration birational to a Weierstrass Model Over a Base with Log Minimal Models

This final lemma establishes that from an elliptic fibration with section it is possible to birationally obtain an elliptic fibration whose canonical divisor admits a Fujita-Zariski decomposition, assuming that the base pair has a log minimal model. This extends Fujita's results in [6] birationally to higher dimensions and sets up the situation to prove a generalized version of Grassi's theorem.

Lemma 4.1.3 (cf. [9, Lemma 1.4])**.** *Let $X \to B$ be a Weierstrass model with the ramification locus having simple normal crossing and B smooth such that (B, Δ) is a Kawamata log terminal pair with Δ a divisor corresponding to $\pi_*\omega_{X/B}$. Then there exists a birationally equivalent fibration $\epsilon : \tilde{X} \to \bar{B}$ such that $(\bar{B}, \bar{\Delta})$ is a log minimal model of* *the log terminal pair* (B, Δ) *and* $K_{\tilde{X}} = \epsilon^*(K_{\bar{B}} + \bar{\Delta}) + \sum c_i \tilde{\pi}^* \Gamma_i + E - G$ *where* $\sum c_i \tilde{\pi}^* \Gamma_i + E - G$ *is effective. In fact, we will have that this is a Fujita-Zariski decomposition of* $K_{\tilde{X}}$.

The proof is similar to the proof of the analogous lemma in Grassi's paper [9, Lemma 1.4] but with adjustments to account for higher dimensions. Using Weierstrass models, we have that there are no multiple fibers. So in Fujita's canonical bundle formula in 2.3.4 we have that all the $m_i = 1$, thus Δ is the divisor associated with $\pi_*(\omega_{X/B})$. To flesh out and clarify this lemma, we have the commutative diagram below:

$$\begin{tikzcd}
 & \tilde{X} \arrow[dl, "\tilde{g}"'] \arrow[dd, "\tilde{\pi}"] \arrow[dddr, "\epsilon"] & \\
X \arrow[dd, "\pi"'] & & \\
 & \tilde{B} \arrow[dl, "g"'] \arrow[dr, "h"] & \\
(B, \Delta) \arrow[rr, dashed, "\psi"] & & (\bar{B}, \bar{\Delta})
\end{tikzcd}$$

where we have that following:

- X is a Weierstrass model over B and $\Delta = \sum_i a_i D_i + \frac{1}{12}\mathbb{J}$ where D_i supports the singular elliptic fibers of π from the Kodaira classification and $a_i \in \mathbb{Q} \cap [0, 1)$ are determined by the Kodaira type from the classification of singular elliptic fibers (2.3.1) and $\mathbb{J}$ is an appropriately chosen divisor corresponding to $J^*\mathcal{O}_{\mathbb{P}^1}(1)$, where J is the j-morphism from $B \to \mathbb{P}^1$.

- $(\bar{B}, \bar{\Delta})$ is the log minimal model of (B, Δ) and $\tilde{B}$ is a common log resolution.

- $\tilde{X}$ is obtained by taking the fiber product of X and $\tilde{B}$ (which will also be a Weierstrass model over $\tilde{B}$) and then resolving the singularities. So $\tilde{\pi} : \tilde{X} \to \tilde{B}$ is a elliptic fibration between smooth projective varieties.

We have that $\pi : X \to B$ is a Weierstrass model, with Δ artificially defined to allow for running the log minimal model. By understanding the singular fibers as orders of vanishing coming from the Weierstrass model/equations, we can understand the difference between $g^*(\Delta)$ and $\tilde{\Delta} = K_{\tilde{X}/\tilde{B}}$. More specifically, we know the resulting singular fibers because the blow ups will increase the order of vanishing of the local Weierstrass equations. For example, the equations

$$y^2 = x^3 + stux + stu$$

is a Weierstrass model over $(s, t, u) \in \mathbb{C}^3$. We know that over a general points of $s = 0$ or $t = 0$ or $u = 0$, we have a type II singular fiber. Yet along the curve $s = t = 0$ we have a type IV singular fiber and over $(0, 0, 0)$, we have a type I_0^* singular fiber. Blowing up the point $(0, 0, 0)$ and pulling back the Weierstrass equations results in a Weierstrass model such that a general point on the exceptional divisor over $(0, 0, 0)$ supports a type I_0^* singular fiber. Comparing this with the discrepancy from Kodaira's classification between $g^*(\Delta)$ to $\tilde{\Delta}$, we observe the difference is modulo an integer.

This is important because we want $K_{\tilde{B}}+\tilde{\Delta}$ to have a Fujita-Zariski decomposition and with theorem 3.2.15 from Birkar we have that $g^*(K_B + \Delta)$ has a Fujita-Zariski decomposition. Since we know that the these two divisors on $\tilde{B}$ differ up to some exceptional divisors, we can actually compute the difference.

Proof. We let $\tilde{\Delta}$ be the divisor supporting the singular fibers of $\tilde{\pi}$ with coefficients coming from the Kodaira classification of singular fibers. For the canonical divisor of $\tilde{B}$, we have that B is smooth and g is a sequence of blow ups. Since (B, Δ) is a *klt* (Kawamata log terminal) pair, we know that given any log resolution, we have:

$$K_{\tilde{B}} = g^*(K_B + \Delta) + \sum_{E_i \text{ any divisor}} a_i E_i$$

$$K_{\tilde{B}} + g_*^{-1}\Delta = g^*(K_B + \Delta) + \sum_{E_i \text{ is } g\text{-exceptional}} a_i E_i \tag{4.1}$$

where $a_i = a(E_i, B, \Delta)$ is the discrepancy of E_i as in [17, Def. 2.22] and $g_*^{-1}\Delta$ is the strict transform of Δ. We work with the second equation because we wish to understand $\tilde{\Delta}$ coming from $\tilde{\pi} : \tilde{X} \to \tilde{B}$ and we know that $Supp(g_*^{-1}\Delta)$ still supports the same singular fibers as Δ since g is a birational morphism. This implies that $g_*^{-1}\Delta \leq \tilde{\Delta}$.

So now we wish to show that "making up the difference" between $\tilde{\Delta}$ and $g_*^{-1}\Delta$ would be enough to put us into the situation of 4.1.1. More concretely, we want:

$$\tilde{\Delta} - g_*^{-1}\Delta + \sum_{E_i \text{ is } g\text{-exceptional}} a_i E_i \geq 0$$

so that adding $\tilde{\Delta} - g_*^{-1}\Delta$ to both sides of the equation 4.1, we have that the difference between $K_{\tilde{B}} + \tilde{\Delta}$ and $g^*(K_B + \Delta)$ is an effective g-exceptional divisor. Now if $a_i \geq 0$, then we are done because $g_*^{-1}\Delta \leq \tilde{\Delta}$ and so the coefficients being added to $\sum_{E_i \text{ is } g\text{-exceptional}} a_i E_i$ are all positive. So adding a positive number to a_i would only make it more positive.

Now if $a_i < 0$ then from [16, Lemma 3.11], we have the following condition on the discrepancy:

$$0 > a_i = a(E_i, B, \Delta) \geq codim(g(E_i)) - 1 - \sum_{g(E_i) \subset \Delta_j} c_j$$

where $\Delta = \sum c_j \Delta_j$ with Δ_i the irreducible components of Δ. So we have that $\sum_{g(E_i) \subset \Delta_j} c_j > codim(g(E_i)) - 1$. Now we know that all the possible coefficients of Δ come from the Kodaira classification from 2.3.1 and since we are working with Weierstrass models they correspond to the order of vanishing of the Weiertrass equations. We have that this codimension equation is only satisfied if $g(E_i)$ is a subset of the intersection of components of Δ. More specifically, if $codim(g(E_i)) = 2$ then we have that $g(E_i) \subset \Delta_{j_k}$ for $k = 1, 2$ and is contained is no other Δ_j since Δ is a simple normal crossing divisor. Similarly if $codim(g(E_i)) = 3$ then we have $g(E_i) \subset \Delta_{j_k}$ for $k = 1, 2, 3$ and is contained is no other Δ_j. So this allows us to focus on the collision points of along Δ on B.

Now let $E := E_i$ with $\alpha = a_i < 0$. This is only true if E maps down to a space whose closure is either curve that is the intersection of two components of Δ or a point that is the intersection of three components of Δ. Now since Δ has simple normal crossing we must have that the closure of the image of E is a smooth component of the intersection of the components of Δ. From [17, Lemma 2.45], we have that there is a sequence of blow ups $f = f_n \circ f_{n-1} \circ ... \circ f_2 \circ f_1 : \hat{B} \to B$ that realize E as a divisor on $\hat{B}$ with discrepancy α since the discrepancy does not depend on the resolution but on the valuation of the function field.

The sequence of blow ups has the property that for each $f_i : B_i \to B_{i-1}$ where $B_0 = B$ and $B_n = \hat{B}$, we have that f_i is the blow up of $Z_{i-1} \subset B_{i-1}$ that contains the image of E on B_{i-1}. Now since Δ is a simple normal crossing divisor with the condition of E and $\alpha < 0$, we must have that f_1 is the blow up of the whole component of the intersection of Δ_i's. Then we have by [17, Lemma 2.29] the following:

$$K_{B_1} + (f_1)_*^{-1}\Delta = f_1^*(K_B + \Delta) + c_1\Gamma_1 \tag{4.2}$$

with Γ_1 containing the image of E on B_1. Now if $c_1 > 0$ then we can add the appropriate term so that the Δ_1 on B_1 supports the singular fibers a Weierstrass model over B_1 that is birational to the Weierstrass model over $\tilde{B}$. If $c_1 < 0$ then using Miranda's approach of viewing singular fibers as orders of vanishing, we will find that the discrepancy corresponding to the exceptional divisor, Γ_1, supporting the "new" singular fibers has coefficients that are the fractional part of the sum of the components of Δ containing $f(E)$. This would give the following bundle formula:

$$K_{B_1} + \tilde{\Delta}_1 = f_1^*(K_B + \Delta)$$

Now we have that (B_1, Δ_1) is a *klt* pair with Δ_1 having simple normal crossings since it is a blow up of the whole intersection of two components of Δ which already has simple normal crossings. Now we can repeat this process of analyzing the exceptional divisor, E, on (B_1, Δ_1). Composing all these together we will get:

$$K_{\hat{B}} + f_*^{-1}(\Delta) = f^*(K_B + \Delta) + \sum_{i=1}^{n-1} c_i\Gamma_i + \alpha E$$

where $\alpha \geq 0$ or we have that $\alpha < 0$ and its negation corresponds to the singular fibers that are supported by a Weierstrass model over E. By lemma 4.1.2 we have fibers over a set dense in E are invariant. Thus if E supports singular fibers then we have that over any other resolution with an exceptional divisor $\tilde{E}$ with the same valuation we will have that $\tilde{E}$ will also have a set dense in $\tilde{E}$ that supports the same singular fibers. So the agreement of the discrepancy on the singular fibers is preserved, thus we have that for the equation 4.1, $a_i < 0$ if and only if E_i supports singular fibers corresponding to $-a_i$.

Now returning to the blow up formula with the log canonical divisor of $(\tilde{B}, \tilde{\Delta})$ we have the following:

$$K_{\tilde{B}} + \tilde{\Delta} = g^*(K_B + \Delta) + \sum_{E_i \text{ is } g\text{-exceptional}} \delta_i E_i$$

where $\delta_i \geq 0$ and $\tilde{\Delta}$ supports the singular fibers of the elliptic fibration $\tilde{\pi} : \tilde{X} \to \tilde{B}$ with appropriate coefficients. By 4.1.1, we have that $K_{\tilde{B}} + \tilde{\Delta}$ admits a Fujita-Zariski decomposition since $g^*(K_B + \Delta)$ admits a Fujita-Zariski decomposition and $\sum_i \delta_i D_i$ is a g-exceptional effective divisor. We will have that in fact the decomposition will be:

$$\begin{aligned}
K_{\tilde{B}} + \tilde{\Delta} &= g^*(K_B + \Delta) + \sum_{E_i \text{ is } g\text{-exceptional}} \delta_i E_i \\
&= h^*(K_{\bar{B}} + \bar{\Delta}) + N + \sum_{E_i \text{ is } g\text{-exceptional}} \delta_i E_i \\
&= h^*(K_{\bar{B}} + \bar{\Delta}) + \sum_i c_i \Gamma_i
\end{aligned}$$

where N is an h-exceptional effective divisor and since E_i is g-exceptional it must also be h-exceptional since $(\bar{B}, \bar{\Delta})$ is a log minimal model. So we combine them to $\sum_i c_i\Gamma_i$ where $c_i \geq 0$ and Γ_i is h-exceptional.

Recall Fujita's canonical bundle formula from 2.3.4 for $\tilde{\pi} : \tilde{X} \to \tilde{B}$. We have that:

$$K_{\tilde{X}} = \tilde{\pi}^*(K_{\tilde{B}} + \tilde{\Delta}) + E - G \tag{4.3}$$

where E is numerically fixed by $\tilde{\pi}^*(K_{\tilde{B}} + \tilde{\Delta})$ and G is mapped via $\tilde{\pi}$ to set of codimension ≥ 2. From Nakayama, [24, Thm 0.2], we have that $E - G$ is effective. Since E is numerically fixed by $\tilde{\pi}^*(K_{\tilde{B}} + \tilde{\Delta})$, we have the following Fujita-Zariski decomposition:

$$K_{\tilde{X}} + G = \tilde{\pi}^*(K_{\tilde{B}} + \tilde{\Delta}) + E$$

Now since $G$is mapped by $\tilde{\pi}$ into a space of dimension $\leq \dim(\tilde{B}) - 2$, by 3.2.13, that $K_{\tilde{X}} = \tilde{\pi}^*(K_{\tilde{B}} + \tilde{\Delta}) + E - G$ admits a Fujita-Zariski decomposition.

To finish the proof, we have that the Fujita-Zariski decomposition $K_{\tilde{B}} + \tilde{\Delta} = h^*(K_{\bar{B}} + \bar{\Delta}) + \sum_i c_i\Gamma_i$. Applying 3.2.11 and substituting into the canonical bundle formula, we will have the explicit Fujtia-Zariski decomposition of $K_{\tilde{X}}$ below:

$$\begin{aligned}
K_{\tilde{X}} &= \tilde{\pi}^*(K_{\tilde{B}} + \tilde{\Delta}) + E - G \\
&= \tilde{\pi}^*(h^*(K_{\bar{B}} + \bar{\Delta}) + \sum_i c_i\Gamma_i) + E - G \\
&= \tilde{\pi}^*(h^*(K_{\bar{B}} + \bar{\Delta})) + \sum_i c_i\tilde{\pi}^*\Gamma_i + E - G \\
&= \epsilon^*(K_{\bar{B}} + \bar{\Delta}) + \sum_i c_i\tilde{\pi}^*\Gamma_i + E - G
\end{aligned}$$

This is the Fujita-Zariski decomposition of $K_{\tilde{X}}$. So we have that $\sum_i c_i\tilde{\pi}^*\Gamma_i + E - G$ is the "negative" effective $\mathbb{Q}$-divisor and so this proves the lemma. ■

4.2 Theorems

4.2.1 On Minimal Models of Elliptic Fourfolds with Section

With the above lemmas, we are in the situation in Grassi's argument in [9], where we have a Fujita-Zariski decomposition and we want to show that the negative portion is contracted when running the relative minimal model program. Adapting arguments from [9], we can do just that with the full power of the Fujita-Zariski decomposition and generalize the results to elliptic fourfolds with section.

Theorem 4.2.1. *Let $\pi : X \to B$ be a Weierstrass model, Δ the divisor associated $\pi_* \mathcal{O}_B(K_{X/B})$ such that (B, Δ) is a Kawamata log terminal threefold with a log minimal model $(\bar{B}, \bar{\Delta})$. Then there exists a birationally equivalent elliptic fibration $\bar{\pi} : \bar{X} \to \bar{B}$, such that $\bar{X}$ is a minimal model of X and $K_{\bar{X}} \equiv \bar{\pi}^*(K_{\bar{B}} + \bar{\Delta})$.*

Proof. We will use notation from lemma 4.1.3, and so we have the following birationallly equivalent elliptic fibrations, $\epsilon : \tilde{X} \to \bar{B}$, with the following formula for the canonical divisor:

$$K_{\tilde{X}} \equiv \tilde{\pi}^*(K_{\tilde{B}} + \tilde{\Delta}) + E - G \tag{4.4}$$

$$K_{\tilde{X}} \equiv \epsilon^*(K_{\bar{B}} + \bar{\Delta}) + \sum_i c_i \tilde{\pi}^* \Gamma_i + E - G \tag{4.5}$$

Now running the relative minimal model program with respect to $\epsilon : \tilde{X} \to \bar{B}$, we then have the following commutative diagram:

$$\begin{tikzcd}
 & & \tilde{X} \arrow[dll, "\tilde{g}"'] \arrow[drr, dashed, "\mu"] \arrow[dd, "\tilde{\pi}"] \arrow[dddrr, "\epsilon"] & & \\
X \arrow[dd, "\pi"'] & & & & \bar{X} \arrow[dd, "\bar{\pi}"] \\
 & & (\tilde{B}, \tilde{\Delta}) \arrow[dll, "g"'] \arrow[drr, "h"] & & \\
(B, \Delta) \arrow[rrrr, dashed, "\psi"] & & & & (\bar{B}, \bar{\Delta})
\end{tikzcd}$$

where we have that $\bar{X}$ is the relative minimal model of $\tilde{X}$ over $\bar{B}$, and we proceed to show that $K_{\bar{X}} \equiv \bar{\pi}^*(K_{\bar{B}} + \bar{\Delta}$ which would imply that $\bar{X}$ is a minimal model.

Consider a common resolution $\hat{X}$ of $\tilde{X}$ and $\bar{X}$, and we have the following diagram:

$$\begin{tikzcd}
 & \hat{X} \arrow[dl, "q"'] \arrow[dr, "p"] & \\
\tilde{X} \arrow[rr, dashed, "\mu"] \arrow[d, "\tilde{\pi}"'] \arrow[dr, "\epsilon"] & & \bar{X} \arrow[dl, "\bar{\pi}"] \\
(\tilde{B}, \tilde{\Delta}) \arrow[r, "h"'] & (\bar{B}, \bar{\Delta}) &
\end{tikzcd}$$

Then from 3.2.15, we have that $q^*(K_{\tilde{X}})$ has the following relative Fujita-Zariski decomposition over $\bar{B}$:

$$q^*(K_{\tilde{X}}) = p^*(K_{\bar{X}}) + F$$

where F is an p-exceptional effective divisor over $\bar{B}$. Substituting with the canonical bundle formula of 4.5 for $K_{\tilde{X}}$ we get the following calculations:

$$\begin{aligned} q^*(K_{\tilde{X}}) &= p^*(K_{\bar{X}}) + F \\ q^*(\epsilon^*(K_{\bar{B}} + \bar{\Delta}) + \sum_i c_i \tilde{\pi}^* \Gamma_i + E - G) &= p^*(K_{\bar{X}}) + F \\ q^*(\epsilon^*(K_{\bar{B}} + \bar{\Delta})) + q^*(\sum_i c_i \tilde{\pi}^* \Gamma_i + E - G) &= p^*(K_{\bar{X}}) + F \\ q^*(\sum_i c_i \tilde{\pi}^* \Gamma_i + E - G) - F &= p^*(K_{\bar{X}}) - q^*(\epsilon^*(K_{\bar{B}} + \bar{\Delta})) \\ q^*(\sum_i c_i \tilde{\pi}^* \Gamma_i + E - G) - F &= p^*(K_{\bar{X}}) - p^*(\bar{\pi}^*(K_{\bar{B}} + \bar{\Delta})) \end{aligned}$$

The last line coming from the fact that the diagram above commutes and $q \circ \epsilon = p \circ \bar{\pi}$. We have that $p^*(\bar{\pi}^*(K_{\bar{B}} + \bar{\Delta}))$ is nef over $\bar{B}$ since it is nef being the pullback of a log canonical divisor of a log minimal model. From the definition of relative Fujita-Zariski decomposition we have that $p^*(K_{\bar{X}}) - p^*(\bar{\pi}^*(K_{\bar{B}} + \bar{\Delta}))$ is effective. So we let:

$$M = q^*(\sum_i c_i \tilde{\pi}^* \Gamma_i + E - G) - F = p^*(K_{\bar{X}}) - p^*(\bar{\pi}^*(K_{\bar{B}} + \bar{\Delta})) \geq 0 \tag{4.6}$$

So we have that $0 \leq M \leq q^*(\sum_i c_i \tilde{\pi}^* \Gamma_i + E - G)$. Recall that $\hat{X} \to \tilde{X}$ is a resolution and $\tilde{X}$ is smooth so we have that:

$$K_{\hat{X}} = q^*(K_{\tilde{X}}) + N$$

where N is an effective exceptional divisor. Using the fact that $\tilde{\pi}$ is an elliptic fibration and substituting with the canonical bundle formula, 4.4 and 4.5, we have the following:

$$\begin{aligned}
K_{\hat{X}} &= q^*(\tilde{\pi}^*(K_{\tilde{B}} + \tilde{\Delta}) + E - G) + N \\
&= q^*(\tilde{\pi}^*(K_{\tilde{B}} + \tilde{\Delta})) + q^*(E - G) + N \\
&= q^*(\epsilon^*(K_{\bar{B}} + \bar{\Delta}) + \sum_i c_i \tilde{\pi}^* \Gamma_i) + q^*(E - G) + N \\
&= q^*(\epsilon^*(K_{\bar{B}} + \bar{\Delta})) + \sum_i c_i q^* \tilde{\pi}^* \Gamma_i + q^*(E - G) + N
\end{aligned}$$

So we have a canonical bundle formula:

$$K_{\hat{X}} = q^*(\epsilon^*(K_{\bar{B}} + \bar{\Delta})) + q^*(\sum_i c_i \tilde{\pi}^* \Gamma_i + E - G) + N$$

We have that by 4.6, $M \leq q^*(\sum_i c_i \tilde{\pi}^* \Gamma_i + E - G)$ in $\hat{X}$. We consider two cases, the first being that $(q \circ \epsilon)(M)$ has codimension 1 and the second case having codimension ≥ 2.

Case 1: We have that the codimension of $(q \circ \epsilon)(M)$ in $\bar{B}$ is 1. This is only possible if $(q \circ \tilde{\pi})(K)$ in $\tilde{B}$ is codimension 1 and has a component which is a effective divisor, D, that is not h-exceptional. We have that from [6], taking two general hyperplane sections on $\tilde{B}$ we get a curve Z that intersects D transversely such that $(q \circ \tilde{\pi})^{-1}(Z) \to Z$ is an elliptic surface and over $x \in Z \cap D$, where $(q \circ \tilde{\pi})^{-1}(x)$ contains a finite collection of exceptional curves with negative intersection matrix. So we have a $(q \circ \epsilon)$-exceptional curve C on $\hat{X}$ such that $C \cdot M = C \cdot C < 0$.

Case 2: We have that the codimension of $(q \circ \epsilon)(M)$ in $\bar{B}$ is ≥ 2. Taking a general hyperplane section H of $\hat{X}$ that intersects M transversly, we let $X_H := \hat{X} \cap H$ and $M_H = M \cap H$. We get a birational morphism $\phi := (q \circ \epsilon)|_{X_H} : X_H \to (q \circ \epsilon)(X_H)$ that contracts K_H. By [4, Lemma 3.6.2], we have that there is a component, F, of M_H that is covered by $(q \circ \epsilon)$-exceptional curves C_i such that $C_i \cdot M_H < 0$. So we have that there is a $(q \circ \epsilon)$-exceptional curve C such that $C \cdot K = C \cdot K_H < 0$.

In both cases, we obtain a $(q \circ \epsilon)$-exceptional curve, C, such that $C \cdot M < 0$. Going back to the formula 4.6, we have:

$$M = p^*(K_{\bar{X}}) - p^*(\bar{\pi}^*(K_{\bar{B}} + \bar{\Delta}))$$
$$C \cdot M = C \cdot p^*(K_{\bar{X}}) - C \cdot p^*(\bar{\pi}^*(K_{\bar{B}} + \bar{\Delta}))$$

But we know that C is exceptional over $\bar{B}$ so that $C \cdot p^*(\bar{\pi}^*(K_{\bar{B}} + \bar{\Delta})) = 0$ and $p^*(K_{\bar{X}})$ is nef so we have:

$$0 > C \cdot M = C \cdot p^*(K_{\bar{X}}) - C \cdot p^*(\bar{\pi}^*(K_{\bar{B}} + \bar{\Delta})) = C \cdot p^*(K_{\bar{X}}) \geq 0$$

which is a contradiction. So we must have that $M = 0$, which implies $q^*(\sum_i c_i \tilde{\pi}^* \Gamma_i + E - G) = F$ so that μ must contract $\sum_i c_i \tilde{\pi}^* \Gamma_i + E - G$. This gives the following formula:

$$p^*(K_{\bar{X}}) - p^*(\bar{\pi}^*(K_{\bar{B}} + \bar{\Delta})) = 0$$
$$p^*(K_{\bar{X}} - \bar{\pi}^*(K_{\bar{B}} + \bar{\Delta})) = 0$$

Pushing forward by p, we then get $K_{\bar{X}} - \bar{\pi}^*(K_{\bar{B}} + \bar{\Delta}) = 0$, which implies $K_{\bar{X}} = \bar{\pi}^*(K_{\bar{B}} + \bar{\Delta})$. Now that $K_{\bar{X}}$ is a pullback of a nef divisor on $\bar{B}$, we have that it is nef and so $\bar{X}$ is not just a relative minimal model but in fact is a minimal model of $\tilde{X}$. ■

4.2.2 On the Canonical Model of Elliptic Fibrations with Section

We will have that lemma 4.1.3, in fact implies a more general theorem on the canonical rings of elliptic fibrations with sections. Specifically we have the following statement:

Theorem 4.2.2. *Let $\pi : X \to B$ be a Weierstrass model, Δ the divisor associated $\pi_* \mathcal{O}_B(K_{X/B})$ such that (B, Δ) is a log pair with a log minimal model $(\bar{B}, \bar{\Delta})$, then canon-*

ical model of X is isomorphic to the log canonical model of $(\bar{B}, \bar{\Delta})$. Equivalently, the canonical ring of X is isomorphic to the log canonical ring of $(\bar{B}, \bar{\Delta})$.

Proof. We have that the canonical model is the projective variety defined by the graded ring:

$$R(X) := \bigoplus_{n=0}^{\infty} H^0(X, nK_X)$$

We have that this is a birational invariant so that if we have $X' \dashrightarrow X$ is a birational map, then we have that $R(X) \cong R(X')$. By [4], we have that the log canonical rings of klt pairs is finitely generated. So we have that the canonical model is well defined in our situation.

With the notation of lemma 4.1.3, we have that $R(X) \cong R(\tilde{X})$. So it is sufficient to work with the canonical ring of $\tilde{X}$. From the definition of Fujita-Zariski decomposition and [6, Cor. 1.9], we have that a Fujita-Zariski decomposition is a CKM-Zariski decomposition. So we will have the following isomorphism:

$$\begin{aligned} H^0(X, nK_{\tilde{X}}) &\cong H^0(X, nK_{\tilde{X}}) \\ &\cong H^0(X, n(\epsilon^*(K_{\bar{B}} + \bar{\Delta}) + \sum_i c_i \tilde{\pi}^* \Gamma_i + E - G)) \\ &\cong H^0(X, n(\epsilon^*(K_{\bar{B}} + \bar{\Delta}))) \end{aligned}$$

We have that by [19, Lemma 2.1.13], we have that $H^0(X, n(\epsilon^*(K_{\bar{B}}+\bar{\Delta}))) \cong H^0(X, n(K_{\bar{B}}+\bar{\Delta}))$. So we have that following isomorphisms

$$H^0(X, nK_{\tilde{X}}) \cong H^0(X, n(K_{\bar{B}} + \bar{\Delta}))$$

for all $n \in \mathbb{N}$. So we have that the canonical ring of $\tilde{X}$ is isomorphic to the log canonical ring of $(\bar{B}, \bar{\Delta})$ which implies the canonical model of X is isomorphic to the log canonical model of $(\bar{B}, \bar{\Delta})$. ■

4.2.3 On Minimal Models of Elliptic Fibrations with Section

The above arguments are general enough so that we have the following corollaries.

Corollary 4.2.3. *Assuming the minimal model program Kawamata log terminal varieties in dimension $n-1$. Let $\pi : X \to B$ be a Weierstrass model, Δ the divisor associated $\pi_* \mathcal{O}_B(K_{X/B})$ such that (B, Δ) is a Kawamata log terminal $(n-1)$-fold with a log minimal model $(\bar{B}, \bar{\Delta})$. Then there exists a birationally equivalent elliptic fibration $\bar{\pi} : \bar{X} \to \bar{B}$, such that $\bar{X}$ is minimal model with at worst terminal singularities and $K_{\bar{X}} \equiv \bar{\pi}^*(K_{\bar{B}} + \bar{\Delta})$.*

Proof. The corollary follows from theorem 4.2.1, but instead of assuming MMP for terminal pairs in dimension n, we use the results of Hacon and Xu from [10, Thm. 2.12] to run the relative minimal model program over the base. The rest of the arguments still holds. ∎

Corollary 4.2.4. *Assuming the minimal model program for log canonical varieties in dimension $n-1$, including the termination of log canonical flips. Let $\pi : X \to B$ be a Weierstrass model, Δ the divisor associated $\pi_* \mathcal{O}_B(K_{X/B})$ such that (B, Δ) is a Kawamata log terminal $(n-1)$-fold with a log minimal model $(\bar{B}, \bar{\Delta})$. Then there exists a birationally equivalent elliptic fibration $\bar{\pi} : \bar{X} \to \bar{B}$, such that $\bar{X}$ is a terminal minimal model and $K_{\bar{X}} \equiv \bar{\pi}^*(K_{\bar{B}} + \bar{\Delta})$.*

Chapter 5

Conclusion

5.1 Concluding Remarks

Theorem 4.2.1 is partial generalization of Grassi's theorem of elliptic threefold. Starting from Grassi's results on minimal models of elliptic threefold and Birkar's results on Zariski decomposition and minimal models, we are able to adapt Grassi's argument to elliptic fourfolds with section and for higher dimensional elliptic fibrations with section assuming the base has a log minimal model. This is done by constructing a birational Weierstrass model using the section and then we proceed to show that the elliptic fibrations structure behaves nicely with lemma 4.1.3 and 4.1.2.

From this birational model of an elliptic fourfold with section by lemma 4.1.1 we are able to establish a Fujita-Zariski decomposition of the canonical divisor of the elliptic fourfold. Adapting Grassi's argument for elliptic threefold and applying the power of a Fujita-Zariski decomposition, we are able to show that the "negative" portion of the Fujita-Zariski decomposition is contracted. This gives theorem 4.2.1 which states that the canonical divisor of the relative minimal model is a pullback of a $\mathbb{Q}$-divisor on the base, which turned out to be the log canonical divisor of a nef threefold pair, implying that the relative minimal model is a minimal model. These arguments will, in fact, generalize to higher dimensions but with the assumption of the log minimal model program for higher dimensions as seen in corollary 4.2.3.

5.2 Future Direction

An immediate question is: Can we weaken the assumption to an elliptic fourfold without section? This seems unlikely in the most general sense because the existence of multiple fibers can potentially obstruct the Fujita-Zariski decomposition. But there is a possibility that these results would hold for elliptic fourfolds without multiple fibers, which is a weaker assumption than elliptic fourfold with section.

A main hurdle with multiple fibers is in part due to a lack of analysis of their behavior in higher dimensions and their birational properties. The calculations and analysis of other singular fibers was simplified for the case of elliptic fibrations with section because of the Weierstrass models that translated the problem into orders of vanishing.

After establishing this relations between minimal models of the total space with the base space, we can ask about the behavior of the fiber structure in relation to the minimal model program. A few immediate question to answer would be: How does the birational properties of a Weierstrass Model interact with the minimal model program? What is the behavior of an elliptic fibration after a flip or a flop on the base? Can we further the results of Grassi in [8] and find conditions to obtaining a minimal model that is equidimensional over the base?

Future work in birational geometry building off this result would be towards understanding algebraic fiber spaces with general fibers having Kodaira dimension 0. Fibrations with general fibers having trivial canonical divisors, like elliptic fibrations, are a special case of this and these spaces are known to factor into the Iitaka fibrations of minimal models. Thus with evidence of the theorem above and properties of the Iitaka fibration, we can hypothesize a more general analog of theorem 4.2.1 proven in this thesis for algebraic fiber spaces with general fibers having Kodaira dimension 0.

Appendix A

Intersection Theory

This section is meant as a review of the necessary background material in intersection theory that allows for the running of the minimal model program. Intersection numbers play a important role for the classical smooth surface case of the minimal model program where the condition of contracting a rational curve, E, is $E \cdot E = -1$, where $E \cdot E$ denotes the self intersection number of E. The formalization of Intersection theory generalized this to smooth varieties in general, but on singular varieties we still have ambiguity. Thus the question of what is meant by $(K_X + \Delta) \cdot C$ for a pair (X, Δ) is a valid question when X is possibly singular.

A.1 Smooth Varieties

A.1.1 Surfaces

Let S be a smooth surface, we then have that prime divisors on S are irreducible curves on S and vice versa. So on S we can simplify to talking about intersection between curves. Now we have that give two curves C and D that are distinct, a naive sense of the number of intersections would be $\#C \cap D$. This is naive in the sense that this only considers the point of intersection but does not account for higher order tangential information. For example, if we have $y = 0$ and $y = x^2$, these two curves intersect only at $(0, 0) \in \mathbb{C}^2$, but we see that the point $(0, 0)$ should be counted twice since x^2 vanishes of order 2 at $(0, 0)$. Thus to capture the proper multiplicities at the point of intersections, we need to account for orders of vanishing, which leads to the following definition:

Definition A.1.1 ([2, Def. I.2])**.** *Let S be a surface with C and D distinct irreducible curves and $x \in C \cap D$. Consider an affine neighborhood $x \in U \subset S$, so that C and D are locally defined in by $f, g \in \mathcal{O}_S(U)$. The the intersection multiplicity of C and D at x is:*

$$m_x(C \cap D) := \dim_{\mathbb{C}} \mathcal{O}_x / \langle f_x, g_x \rangle$$

where $\mathcal{O}_x$ is the local ring of x in S and f_x, g_x is taken to be the image of f, g from the canonical map $\mathcal{O}_S(U) \hookrightarrow \mathcal{O}_x$.

In this way, we have that the intersection number of C and D can be defined as:

Definition A.1.2 ([2, Def. I.3])**.** *Let C and D be distinct irreducible curves in S, then the intersection number of C and D is defined:*

$$C \cdot D = \sum_{x \in C \cap D} m_x(C \cap D)$$

Thus we are able to define the intersection number of two distinct irreducible curves. Since curves are divisors, we can extend this definition of intersection linearly, but to properly extend the definition we will need a reasonable definition of a *self intersection* of a curve C. It will turn out that the extension is via line bundles and sheaves. If we analyze the intersection number $C \cdot D$, we will have that if D and D' are linearly equivalent then $C \cdot D = C \cdot D'$. Thus there is a invariance under linear equivalence. This implies that this definition of intersection can be extended to line bundles of S. To shorten a long story, we will have:

Lemma A.1.3 ([2, cf. Lemma. I.6])**.** *Let C and D be irreducible curves on S, then we* *have that:*

$$C \cdot D = \deg\left(\mathcal{O}_S(C)|_D\right)$$

where $\mathcal{O}_S(C)|_D$ is the line bundle $\mathcal{O}_S(C)$ restricted to the curve D.

This lemma allows us to extend the definition of intersection of curves to S. Additionally this allows us to define the self intersection number of C to be $\mathcal{O}_S(C)|_C$, thus allowing for even negative "self intersections". While this may seem strange it is best to view this as an invariant associated with the curve C on the surface S. This can be viewed as that C has no curves linearly equivalent to it other than itself, which is a feature of some Zariski decompositions and also exceptional divisors.

A.1.2 Higher Dimensions

The case of smooth surfaces actually opens the way for approaching intersection theory in higher dimensions, at least at the level of intersecting divisors and curves. Now let X be a smooth variety and C, a curve on X, and D, a divisor on X. We need a definition of $C \cdot D$ in the case X is smooth that would allow for the running of the minimal model program. The proper definition, going through Hilbert polynomials and Euler characteristics that go towards defining the intersection between C and D can be found in [5, Sec. 1.2]. We will have that the definition will be:

Definition A.1.4. *Let X be a smooth variety with C an irreducible curve and D a divisor on X. Then we have the intersection number of C and D is defined as:*

$$C \cdot D = \deg\left(\mathcal{O}_S(D)|_C\right)$$

A.2 Projection Formula

The projection formula is a means to describe the relation between intersection theory and morphisms. More specifically, given a proper morphism $f : X \to Y$, we can understand some of the intersections on X through Y and vice versa. This is important for us mainly because, working with fiber spaces and elliptic fibrations, we are comparing the intersections of divisors and curves on the base of the fiber spaces to that of the total spaces. The source of the general case of the projection formula can be found in [5, Prop. 1.10].

Let $f : X \to Y$ be a proper morphism of smooth varieties. Let C be a curve on X. Define $\pi_*(C)$ to be 0 if π contracts C, otherwise define $\pi_*(C) := d\,\pi(C)$ where $d = \deg(C \to \pi(C))$ is the degree of π restricted to C. The **projection formula** is given by:

$$\pi^*(D) \cdot C = D \cdot \pi_*(C)$$

where D is a Cartier divisor on Y. In fact this an extension of the definition of the intersection number as we can consider the morphism $C \hookrightarrow X$ and we are taking the pullback the divisor D (more properly the line bundle associated to D) onto C. One important aspect, is that given a subvariety $X \subset Y$, we have that $X \hookrightarrow Y$ so that the intersection on X of curves on X and divisors on Y restricted to X agrees as intersections on Y.

A.3 Intersection Theory on Pairs

For the results, we have (X, Δ) is a pair with at worst potentially log canonical singularities. For singular varieties, the version of intersection theory above breaks down since we require line bundles, or in other words, we needed our divisor to be Cartier. Imposing this condition on our varieties is a very strong condition that would be too strict to run the minimal model program. It is because of this that we have the property that our varieties be $\mathbb{Q}$-factorial, so that given and divisor D on X we will have that some non-zero integer multiple of D is a Cartier divisor on X.

Now we can extend our definition of intersection to $\mathbb{Q}$-Cartier divisors. Let C be an irreducible curve on X and D a $\mathbb{Q}$-divisor, then for some $m > 0$ we will have that mD is Cartier. Then we will have:

$$C \cdot (mD) = n = \deg\left(\mathcal{O}_S(mD)|_C\right) \in \mathbb{Z}$$

Then by formally dividing by m, we define $C \cdot D = \frac{n}{m} \in \mathbb{Q}$. Thus having $\mathbb{Q}$-Cartier divisors results in rational intersection numbers.

Part of the difficulties of running the log minimal model program on pairs (X, Δ) where X is $\mathbb{Q}$-factorial, was the fact that small contractions resulted in a variety that was no longer $\mathbb{Q}$-factorial. In fact, the resulting variety's canonical divisor was no longer $\mathbb{Q}$-Cartier, thus the resulting variety was *too singular* to run the MMP. So the procedure of flips was necessary as a means to get around this problem.

Appendix B

Related Results

B.1 Simpler Proof of Weaker Statement

Lemma B.1.1. *Let $(X/Z, \Lambda)$ be a log pair with $(\bar{X}/Z, \bar{\Lambda})$ a relative minimal model obtained from running MMP on $(X/Z, \Lambda)$ and $\mu : X \dashrightarrow \bar{X}$ the sequence of contractions and flips. Assume that $K_X + \Lambda = P + N$ is a Fujita-Zariski decomposition over Z of the log canonical divisor of (X, Λ), then $\mu_* N \equiv 0$. In other words, N is contracted in the process of running the relative log MMP.*

Proof. Take a common log resolution of (X, Λ) and $(\bar{X}, \bar{\Lambda})$ which gives the following diagram:

$$\begin{tikzcd} & \tilde{X} \arrow[dl, "g"'] \arrow[dr, "h"] & \\ (X, \Lambda) \arrow[rr, dashed, "\mu"] \arrow[dr, "f"'] & & (\bar{X}, \bar{\Lambda}) \arrow[dl, "\bar{f}"] \\ & Z & \end{tikzcd}$$

From [3], we have that:

$$g^*(K_X + \Lambda) = h^*(K_{\bar{X}} + \bar{\Lambda}) + E$$

for an effective h-exceptional divisor E and that this is a Fujita-Zariski decomposition of

$g^*(K_X + \Lambda)$ over Z. Specifically, we have that $h^*(K_{\bar{X}})$ is nef over Z and E is effective.

From the assumptions, we have that $K_X+\Lambda = P+N$ is a Fujita-Zariski decomposition over Z and so from the definition of Fujita-Zariski decomposition we have that $h^*(K_{\bar{X}}) \leq g^*(P)$, which means:

$$\begin{aligned} g^*(K_X + \Lambda) &= g^*(P) + g^*(N) \\ &= h^*(K_{\bar{X}} + E \\ g^*(P) - h^*(K_{\bar{X}} &= E - g^*(N) \end{aligned}$$

This gives $g^*(N) \leq E$. But we know that N' is contracted by h and so $g^*(N)$ is also contracted by h. Since push forwards commute in the above commutative diagram, this implies that $\mu_* N = h_*(g^*(N)) \equiv 0$. ■

Theorem B.1.2. *Let $\pi : X \to B$ be a Weierstrass model, Δ the divisor associated $\pi_*\mathcal{O}_B(K_{X/B})$ such that (B, Δ) is a log terminal threefold with a log minimal model $(\bar{B}, \bar{\Delta})$. Then there exists a birationally equivalent rational elliptic fibration $\bar{\pi} : \bar{X} \dashrightarrow \bar{B}$, such that $\bar{X}$ is a minimal model of X and $K_{\bar{X}} \equiv \bar{\pi}^*(K_{\bar{B}} + \bar{\Delta})$.*

Proof. We use the notation of lemma 4.1.3, so we have that there is a birationally equivalent fibration $\epsilon : \tilde{X} \to \bar{B}$, with the following formula of the canonical divisor:

$$K_{\tilde{X}} \equiv \epsilon^*(K_{\bar{B}} + \bar{\Delta}) + \sum_i c_i \tilde{\pi}^* \Gamma_i + E - G$$

which is a Fujita-Zariski decomposition of $K_{\tilde{X}}$. Running the minimal model program on $\tilde{X}$, we obtain a minimal model $\bar{X}$ with a birational map $\phi : \tilde{X} \dashrightarrow \bar{X}$, with a rational map $\bar{\pi} : \bar{X} \dashrightarrow \bar{B}$.

Now since $\bar{\pi}$ is a rational map, the pullback map on divisors is not very well defined but we know that $\bar{\pi}$ is defined in the compliment of the space of codimension ≥ 2 and it's image has compliment of codimension ≥ 2 on $\bar{B}$. This is true since $\bar{\pi}$ is not defined on the image of the contracted divisors of $\tilde{X}$, so it is defined in codimension 1. By lemma B.1.1, we have that $\sum_i c_i \tilde{\pi}^* \Gamma_i + E - G$ is contracted by $\bar{\pi}$ since it is the negative part of a Fujita-Zariski decomposition. So on the base, $\bar{B}$, these divisors on $\tilde{X}$ is supported on h-excpetional divisors on $\tilde{B}$ that is contracted. So we have that $\bar{\pi} : \bar{X} \to \bar{B}$ is defined in codimension 1 and has image in $\bar{B}$ with complement of codimension ≥ 2.

So given a divisor, D, on $\bar{B}$, we can look at the preimage closure, $\overline{\bar{\pi}^{-1}(D)}$ in $\bar{X}$, which would be a map from $\mathbb{Q}$-divisors on $\bar{B}$ to $\mathbb{Q}$-divisors on $\bar{X}$. This is well defined since $\bar{\pi}$ is defined in codimension 1 and has image with complement of codimension ≥ 2. Calling this the pullback map and analyzing the commutative diagram, we have that $K_{\bar{X}} \equiv \bar{\pi}^*(K_{\bar{B}} + \bar{\Delta})$. ∎

B.2 Divisors Covered by $K_X + \Delta$-negative curves

Proposition B.2.1. *Let $\mu : X \dashrightarrow X^+$ be a flip where $\dim X = n \geq 3$. Let D be an irreducible divisor such that there is a family of curves $\{C_\gamma\}$ that densely covers D such that $K_X \cdot C_\gamma < 0$, then there exists an irreducible divisor D^+ on X^+ such that D^+ is densely covered by $\{C^+_\beta\}$ and $K_{X^+} \cdot C^+_\beta < 0$, where $\{C^+_\beta\}$ strict transform of an infinite subfamily of $\{C_\gamma\}$.*

Proof. The proof is essentially the same as Grassi's proof in [9] with adjustments for generalizing to higher dimensions. Let $\hat{X}$ be a common resolution of X and X^+. Let $A \subset X$ be the exceptional locus of ϕ (and μ). Let E_i denote the exceptional divisor on

$\hat{X}$ of g and h. So we have the following diagram:

$$\begin{tikzcd} & \hat{X} \arrow[dl, "g"'] \arrow[dr, "h"] & \\ X \arrow[rr, dashed, "\mu"] \arrow[dr, "\phi"'] & & X^+ \arrow[dl, "\phi^+"] \\ & Z & \end{tikzcd}$$

We have that μ is an isomorphism of codimension 1 so that g, h are isomorphism outside the inverse image of A. In addition with the negativity lemma, [17, Lemma 3.38], we have that:

$$g^*(K_X) = h^*(K_{X^+}) + \sum_i (a_i^+ - a_i)E_i$$

where $(a_i^+ - a_i) \geq 0$ and $(a_i^+ - a_i) > 0$ if and only if $g(E_i) \subset A$ via the negativity lemma.

Now there exists a subfamily $\{C_\beta\}$ of $\{C_\gamma\}$ such that C_β is not contained in A since a curve in $\{C_\gamma\}$ is contained in A if and only if it is part of the numerical equivalence class of the curve corresponding to the extremal contraction that induced the flipping contraction. But we have that the codimension of A is ≥ 2 and $\{C_\gamma\}$ densely covers a codimension 1 space. This also shows that $\{C_\beta\}$ still densly covers D. Since $\hat{X}$ is obtained via a finite number of blow ups of codimension ≥ 2 spaces we can omit curves contained in those locus of blow ups also.

With all these choices made we have that $\{C_\beta\}$ densely covers D and we have that the family of strict transform, $\{\hat{C}_\beta\}$, on $\hat{X}$ is such that any curves $\hat{C}_\beta$ is not contained in any of the exceptional divisors E_i. Thus we have that $E_i \cdot \hat{C}_\beta \geq 0$. So we have the following:

$$0 > K_X \cdot C_\beta = g^*(K_X) \cdot \hat{C}_\beta = h^*(K_{X^+}) \cdot \hat{C}_\beta + \sum_i (a_i^+ - a_i)E_i \cdot \hat{C}_\beta$$

This gives that $0 > h^*(K_{X^+}) \cdot \hat{C}_\beta$ but we know via the projection formula that $h^*(K_{X^+}) \cdot$

$\hat{C}_\beta = K_{X^+} \cdot h_*\hat{C}_\beta$, so we obtain curves $h_*\hat{C}_\beta$ such that $K_{X^+} \cdot h_*\hat{C}_\beta < 0$. Now we know that $\{h_*\hat{C}_\beta\}$ are just the strict transform of $\{C_\beta\}$ via μ. Since μ is the isomorphism of codimension 1, we know that if we let $D^+ = \mu_* D$ we would have that $\{h_*\hat{C}_\beta\}$ will densly cover D^+ since $\{C_\beta\}$ dense covers D. So we let $C_\beta^+ := h_*\hat{C}_\beta$ and then we have a family of curves $\{C_\beta^+\}$ on X^+ that densely covers an irreducible divisor D^+.

■

Proposition B.2.2. *Let $\mu : X \to X'$ be a divisorial contraction of $\mathbb{Q}$-factorial terminal varieties where $\dim X = n \geq 3$. Let D be an irreducible divisor such that there is a family of curves $\{C_\gamma\}$ that densely covers D such that $K_X \cdot C_\gamma < 0$ and μ does not contract D then there exists an irreducible divisor $\hat{D}$ on X' such that $\hat{D}$ is densely covered by $\{\hat{C}_\beta\}$ and $K_{X'} \cdot \hat{C}_\beta < 0$, where $\hat{C}_\beta$ is the strict transform of an infinite subfamily of $\{C_\gamma\}$.*

Proof. We have that $K_X = \mu^*(K_{X'}) + E$, where E is the effective exceptional divisor. Since D is not contracted by μ we have none of the curve of $\{C_\gamma\}$ is contracted and their intersection with E is non-negative. Let $\{\hat{C}_\beta\}$ be the strict transform of $\{C_\gamma\}$, then we have the followng:

$$\hat{C}_\beta \cdot K_{X'} = C_\beta \cdot \mu^*(K_{X'}) = C_\beta \cdot (K_X - E) = C_\beta \cdot K_X - C_\beta \cdot E < 0$$

So we let $\hat{D}$ be the image of D and we have that $\hat{D}$ is densely covered by $K_{X'}$-negative curves.

■

Bibliography

[1] Lucian Bădescu. *Algebraic surfaces*. Universitext. Translated from the 1981 Romanian original by Vladimir Maşek and revised by the author. Springer-Verlag, New York, 2001, pp. xii+258. ISBN: 0-387-98668-5. DOI: 10.1007/978-1-4757-3512-3. URL: https://doi.org/10.1007/978-1-4757-3512-3.

[2] Arnaud Beauville. *Complex algebraic surfaces*. Second. Vol. 34. London Mathematical Society Student Texts. Translated from the 1978 French original by R. Barlow, with assistance from N. I. Shepherd-Barron and M. Reid. Cambridge University Press, Cambridge, 1996, pp. x+132. ISBN: 0-521-49510-5; 0-521-49842-2. DOI: 10.1017/CBO9780511623936. URL: https://doi.org/10.1017/CBO9780511623936.

[3] Caucher Birkar. "On existence of log minimal models and weak Zariski decompositions". In: *Math. Ann.* 354.2 (2012), pp. 787–799. ISSN: 0025-5831. DOI: 10.1007/s00208-011-0756-y. URL: http://doi.org/10.1007/s00208-011-0756-y.

[4] Caucher Birkar et al. "Existence of minimal models for varieties of log general type". In: *J. Amer. Math. Soc.* 23.2 (2010), pp. 405–468. ISSN: 0894-0347. DOI: 10.1090/S0894-0347-09-00649-3. URL: https://doi.org/10.1090/S0894-0347-09-00649-3.

[5] Olivier Debarre. *Higher-dimensional algebraic geometry*. Universitext. Springer-Verlag, New York, 2001, pp. xiv+233. ISBN: 0-387-95227-6. DOI: 10.1007/978-1-4757-5406-3. URL: https://doi.org/10.1007/978-1-4757-5406-3.

[6] Takao Fujita. “Zariski decomposition and canonical rings of elliptic threefolds”. In: *J. Math. Soc. Japan* 38.1 (1986), pp. 19–37. ISSN: 0025-5645. DOI: 10.2969/jmsj/03810019. URL: http://doi.org/10.2969/jmsj/03810019.

[7] Yoshinori Gongyo and Brian Lehmann. “Reduction maps and minimal model theory”. In: *Compos. Math.* 149.2 (2013), pp. 295–308. ISSN: 0010-437X. DOI: 10.1112/S0010437X12000553. URL: https://doi.org/10.1112/S0010437X12000553.

[8] Antonella Grassi. “Log contractions and equidimensional models of elliptic threefolds”. In: *J. Algebraic Geom.* 4.2 (1995), pp. 255–276. ISSN: 1056-3911.

[9] Antonella Grassi. “On minimal models of elliptic threefolds”. In: *Math. Ann.* 290.2 (1991), pp. 287–301. ISSN: 0025-5831. DOI: 10.1007/BF01459246. URL: https://doi.org/10.1007/BF01459246.

[10] Christopher D. Hacon and Chenyang Xu. “Existence of log canonical closures”. In: *Invent. Math.* 192.1 (2013), pp. 161–195. ISSN: 0020-9910. DOI: 10.1007/s00222-012-0409-0. URL: https://doi.org/10.1007/s00222-012-0409-0.

[11] Robin Hartshorne. *Algebraic geometry.* Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977, pp. xvi+496. ISBN: 0-387-90244-9.

[12] Heisuke Hironaka. “Resolution of singularities of an algebraic variety over a field of characteristic zero. I, II”. In: *Ann. of Math. (2)* **79** *(1964), 109–203; ibid. (2)* 79 (1964), pp. 205–326. ISSN: 0003-486X.

[13] Yujiro Kawamata. “Flops connect minimal models”. In: *Publ. Res. Inst. Math. Sci.* 44.2 (2008), pp. 419–423. ISSN: 0034-5318. DOI: 10.2977/prims/1210167332. URL: https://doi.org/10.2977/prims/1210167332.

[14] Yujiro Kawamata. “Kodaira dimension of certain algebraic fiber spaces”. In: *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* 30.1 (1983), pp. 1–24. ISSN: 0040-8980.

[15] Kunihiko Kodaira. “On compact analytic surfaces. II, III”. In: *Ann. of Math. (2) 77 (1963), 563–626; ibid.* 78 (1963), pp. 1–40. ISSN: 0003-486X.

[16] János Kollár. "Singularities of pairs". In: *Algebraic geometry—Santa Cruz 1995.* Vol. 62. Proc. Sympos. Pure Math. Amer. Math. Soc., Providence, RI, 1997, pp. 221–287.

[17] János Kollár and Shigefumi Mori. *Birational geometry of algebraic varieties.* Vol. 134. Cambridge Tracts in Mathematics. With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original. Cambridge University Press, Cambridge, 1998, pp. viii+254. ISBN: 0-521-63277-3. DOI: 10.1017/CBO9780511662560. URL: http://doi.org/10.1017/CBO9780511662560.

[18] Ching-Jui Lai. "Varieties fibered by good minimal models". In: *Math. Ann.* 350.3 (2011), pp. 533–547. ISSN: 0025-5831. DOI: 10.1007/s00208-010-0574-7. URL: https://doi.org/10.1007/s00208-010-0574-7.

[19] Robert Lazarsfeld. *Positivity in algebraic geometry. I.* Vol. 48. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Classical setting: line bundles and linear series. Springer-Verlag, Berlin, 2004, pp. xviii+387. ISBN: 3-540-22533-1. DOI: 10.1007/978-3-642-18808-4. URL: https://doi.org/10.1007/978-3-642-18808-4.

[20] Rick Miranda. "Smooth models for elliptic threefolds". In: *The birational geometry of degenerations (Cambridge, Mass., 1981).* Vol. 29. Progr. Math. Birkhäuser, Boston, Mass., 1983, pp. 85–133.

[21] Joaquín Moraga. "Termination of pseudo-effective 4-fold flips". In: *ArXiv e-prints* (Feb. 2018). arXiv: 1802.10202 [math.AG].

[22] Shigefumi Mori. "Flip theorem and the existence of minimal models for 3-folds". In: *J. Amer. Math. Soc.* 1.1 (1988), pp. 117–253. ISSN: 0894-0347. DOI: 10.2307/1990969. URL: https://doi.org/10.2307/1990969.

[23] Shigefumi Mori. "Threefolds whose canonical bundles are not numerically effective". In: *Ann. of Math. (2)* 116.1 (1982), pp. 133–176. ISSN: 0003-486X. DOI: 10.2307/2007050. URL: https://doi.org/10.2307/2007050.

[24] Noboru Nakayama. "On Weierstrass models". In: *Algebraic geometry and commutative algebra, Vol. II.* Kinokuniya, Tokyo, 1988, pp. 405–431.

[25] Jaroslaw Wlodarczyk. "Simple constructive weak factorization". In: *Algebraic geometry—Seattle 2005. Part 2.* Vol. 80. Proc. Sympos. Pure Math. Amer. Math. Soc., Providence, RI, 2009, pp. 957–1004. DOI: 10.1090/pspum/080.2/2483958. URL: http://doi.org/10.1090/pspum/080.2/2483958.

[26] Oscar Zariski. "The theorem of Riemann-Roch for high multiples of an effective divisor on an algebraic surface". In: *Ann. of Math. (2)* 76 (1962), pp. 560–615. ISSN: 0003-486X. DOI: 10.2307/1970376. URL: https://doi.org/10.2307/1970376.